Vortex velocity pair correlations

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The vortex velocity probability distribution for two distinct vortices is determined for the case of phase-ordering kinetics in systems with point defects. The n-vector model driven by time-dependent Ginzburg-Landau dynamics for a nonconserved order parameter is considered. The description includes the effects of other vortices and order-parameter fluctuations. At short distances the most probable configuration is that a vortex-antivortex pair has only a nonzero relative velocity that is inversely proportional to the distance between them. The coefficient of proportionality is determined explicitly. [S1063-651X(97)08909-5]

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I. INTRODUCTION

It seems plausible that much of the structure one sees in the phase ordering of many materials [1,2] can be associated with the evolution and correlation among defects [3] such as vortices, monopoles, and disclinations. These topologically robust objects grow out of the frustration suffered by a system with a continuous symmetry that is thermodynamically driven to align in a broken symmetry state. In the case of the n-vector model with the number of components (n) of the order parameter equal to the spatial dimensionality (d) one has point defects that are vortices for $n=2$ and monopoles for $n=3$. Because of the conservation of topological charge, the ordering in these systems is through the charge conserving process of vortex-antivortex annihilation. Topological constraints render the ordering in such systems to be largely independent of the microscopic details of the material. In this paper the following question is addressed: What is the probability, given a vortex at position $\vec{r}_1$ with velocity $\vec{v}_1$, that one will find a vortex at position $\vec{r}_2$ with velocity $\vec{v}_2$? Clearly, in answering this question we obtain a tremendous amount of information about the dynamics of vortices.

The calculation of the two-vortex velocity probability distribution is a very involved process. In principle, one could probe vortex dynamics by applying a force. Unfortunately, in neutral systems it is very difficult to couple directly to the vortices. The two-vortex velocity probability distribution serves this purpose by looking at the motion of one vortex in the fixed presence of another vortex a known distance away.

The physical results of this calculation, carried out in detail for $n=d=2$, are relatively simple to state. The appropriate probability distribution is a function only of the scaled velocities $\vec{u}_i=\vec{v}_i/\overline{v}$ for $i=1$ or 2 and the scaled separation $\vec{x}=(\vec{r}_1-\vec{r}_2)/L(t)$. Here $L(t)$ is the characteristic length in the problem, which grows with time t after a quench as $t^{1/2}$ in the present case [4] and drives the scaling behavior [1] found in the problem. The characteristic velocity $\overline{v}$, defined carefully below, is inversely proportional to $L(t)$. For a given scaled separation x between two chosen vortices, the most probable configuration corresponds, as expected, to a state with zero total momentum and a nonzero relative momentum only along the axis connecting the vortices

$$\vec{v}_1=-\vec{v}_2=v\hat{x}. \tag{1}$$

Moreover, there is a definite nonzero value for $v=v_{max}$ for a given value of x. These most probable values are given as a function of x in Fig. 1. The most striking feature of these results is that for small x the most probable velocity goes as

$$v_{max}=\frac{\kappa}{R}, \tag{2}$$

where R is the unscaled separation between the vortices and $\kappa=2.19$ in dimensionless units defined below. The result giving v_{max} inversely proportional to R is consistent with overdamped dynamics where the relative velocity of the two vortices is proportional to the force, which in turn is the derivative of a potential that is logarithmic in the separation distance. Thus these most probable results are consistent with the short-distance behavior being dominated by the annihilation of vortex-antivortex pairs. From previous work [5] we know that there is low probability of finding like-signed vortices at short distances. Thus our results giving the velocity as a function of separation should be interpreted in terms of annihilating vortex-antivortex pairs. The results for same-

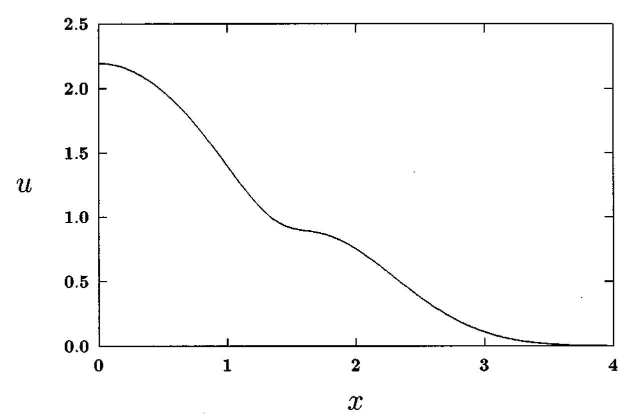

FIG. 1. Most probable scaled velocity of vortex 2 multiplied by the magnitude of the scaled distance of separation x, $u=\vec{x}\cdot\vec{u}(2)L/\Gamma c$ versus x. This velocity is directed along $\hat{x}$, the line connecting the two tagged vortices. The most probable velocity of vortex 1 is equal and opposite to that of vortex 2.

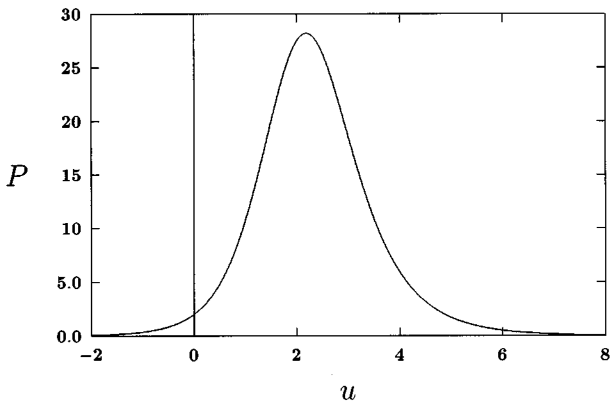

FIG. 2. Probability distribution (unnormalized) for the scaled longitudinal velocity $u=\vec{x}\cdot\vec{u}(2)L/\Gamma c=-\vec{x}\cdot\vec{u}(1)L/\Gamma c$ in the small-scaled-distance x limit. The peak in this curve gives the small-x limit of the quantity plotted in Fig. 1.

signed vortices can also be carried out, but is considerably more involved, as discussed below.

The work here builds on the work in Ref. [6], where the single-vortex velocity distribution was determined. As in the single-vortex case, there are significant widths associated with these most probable results. The widths come about because of the existence of other vortices as well as fluctuations in the order-parameter field. There are, as shown in Fig. 2, significant widths in the probabilities due to the presence of other vortices and fluctuations in the order-parameter field. For example, as $x\rightarrow 0$, while the most probable relative velocity is $4.38/R$, the half-width at half maximum for this quantity, in these same units, is $2.08/R$. In the large separation limit the probabilities become, as expected, uncorrelated and each has the distribution of velocities found previously [6] for a single vortex.

The analysis here is built upon previous work on the ordering kinetic of $O(n)$ symmetric systems. The best available theories [2] for the order parameter correlation function were built up in the early 1990s and have led to the belief that we have a fairly good understanding of how to calculate the associated scaling function. It also has become clear that the order-parameter correlation function or structure factor is a rather structureless quantity that does not give a great deal of direct information about the underlying disordering agents. This led Liu and Mazenko [7] to look directly at the correlations between defects in the scaling regime. The key different element in this work, as discussed in some detail below, was the realization that the positions of the vortices could be labeled by the zeros of the order parameter field, which could in turn be mapped onto the zeros of an auxiliary field $\vec{m}(\vec{x},t)$. They were able to show, following work by Halperin [8], how one could write explicit expressions for the signed and unsigned vortex densities in terms of the auxiliary field $\vec{m}(\vec{r},t)$. This then avoids the technically defeating step normally encountered, which requires one to identify the vortex positions.

The signed vortex density correlation function was determined analytically in Ref. [7] in terms of the variance of the auxiliary field under the assumption the auxiliary field is Gaussian. This calculation left the auxiliary field correlation function $f(x)$ undetermined. Liu and Mazenko assumed that one could use $f(x)$ determined from a treatment of the order-parameter dynamics *away from the defect cores*.

The charged or *signed* vortex autocorrelation function does not separate out all of the desired information since it mixes the correlations between like- and unlike-signed vortices. It is not difficult to introduce an *unsigned* vortex autocorrelation function. Between the signed and unsigned autocorrelation functions one can construct linear combinations that give the vortex-vortex and vortex-antivortex correlation functions. Unfortunately, for technical reasons it is more difficult to determine the uncharged autocorrelation function. Only recently have these difficulties been overcome by Mazenko and Wickham [5]. They found the results, expected on physical grounds, that there is a depletion zone at short distances for the vortex-vortex correlation function indicating repulsion. Simulations [9] and experiments [10] also show a depletion zone at short distances for like-signed defects. This is expected on physical grounds since like-signed defects repel one another. There is a clear discrepancy between theory and simulation results at short-scaled distances. The theory shows a monotonic behavior as the separation distance goes to zero. The simulation, however, shows a maximum at short separation distances and then falls rapidly to zero. The depletion zone seen in this case in the simulations is harder to understand physically since the pair is attractive and headed toward annihilation. While the theory satisfies the sum rule implied by topological charge conservation, it does not appear that this general constraint is satisfied by the simulations. It appears that the short-distance behavior in the simulations is contaminated by the choice of a vortex core distance that is comparable to distances associated with the unphysical depletion zone.

It seems clear that it would be desirable to supplement this information on the spatial correlation of vortices with information concerning vortex velocities. It was recently shown by the present author [6] that one could write down an explicit expression for the velocities associated with point defects in terms of the order-parameter field. A key ingredient in this development is the identification of a continuity equation satisfied by the signed or charged vortex density. This continuity equation gives a fundamental expression for conservation of topological charge in the system. Using the Gaussian closure assumption, one can determine the single-vortex velocity distribution $P[\vec{v}_1]$. The most interesting physical result is that there is a large velocity tail that was interpreted there as arising from the high velocities in the late stages of vortex-antivortex annihilation. Bray [11] has used scaling arguments to obtain the same large velocity tail. The existence of these large velocities will be supported by the calculation carried out here.

One common and concerning element in the calculations of defect correlation functions and defect velocity distributions is the requirement that the auxiliary field scaled correlation function $f(x)$ be analytic as a function of x for short scaled distances. For example, the fourth-order gradient $[-\nabla_x^4 f(x)]|_{x=0}$ enters naturally into the analysis of $P[\vec{v}_1]$. The need for analyticity in x for $f(x)$ is not naturally consistent with the simplest self-consistent analysis of $f(x)$ following a treatment of the order-parameter correlation function. Mazenko and Wickham [12] recently showed that one can construct the theory so that $f(x)$ is analytic in x for small x, but this was at the expense of making the properties of the order-parameter correlation function worse [13].

The tension between using the order-parameter dynamics to determine $f(x)$ and the requirement that $f(x)$ be analytic in order to treat defect dynamics has been, to a degree, relieved by the very recent work of Mazenko and Wickham [14]. They used the recently proposed continuity equation for topological charge to derive the equation satisfied by the auxiliary field correlation function under the circumstances that the field is constrained to be near a defect core. As discussed briefly in Sec. III C of this paper, they find the clean result that the auxiliary field correlation function determined in this manner satisfies a linear equation. This result is self-consistent with the assumption that the auxiliary field is Gaussian. The solution of the associated linear equation has the Ohta-Jasnow-Kawasaki (OJK) [15] form

$$f(x)=e^{-(1/2)x^2}, \tag{3}$$

which is clearly analytic in the small-x regime. They argue in Ref. [14] that the use of the Gaussian assumption in determining defect dynamics, such as $P(\vec{v}_1)$, has a stronger fundamental justification than in the case of the determination of the order-parameter correlation function. In the calculation of the two-vortex velocity probability distribution presented here it is assumed that the order parameter field can be replaced by a Gaussian field in those portions of space near a vortex core and the associated auxiliary field correlation function is of the OJK form.

II. ORDER-PARAMETER DYNAMICS

The system studied here has a defect dynamics generated by the time-dependent Ginzburg-Landau model satisfied by a nonconserved n-component vector order parameter $\vec{\psi}(\vec{r},t)$:

$$\frac{\partial \vec{\psi}}{\partial t}=\vec{K}\equiv-\Gamma\frac{\delta F}{\delta \vec{\psi}}+\vec{\eta}, \tag{4}$$

where Γ is a kinetic coefficient and F is a Ginzburg-Landau effective free energy assumed to be of the form

$$F=\int d^dr\left(\frac{c}{2}(\nabla\vec{\psi})^2+V(|\vec{\psi}|)\right), \tag{5}$$

where $c>0$ and the potential V is assumed to be of the $O(n)$ symmetric degenerate double-well form. Since only these properties of V will be important in what follows we need not be more specific in our choice for V [16]. $\vec{\eta}$ is a thermal noise that is related to Γ by a fluctuation-dissipation theorem. We assume that the quench is from a high temperature $(T_I>T_c)$, where the system is disordered to zero temperature where the noise can be set to zero $(\vec{\eta}=\vec{0})$. It is believed that our final results are independent of the exact nature of the initial state, provided it is a disordered state.

It is well established that for late times following a quench from the disordered to the ordered phase the dynamics obey scaling and the system can be described in terms of a single growing length $L(t)$, which is characteristic of the spacing between defects. In this scaling regime the order-parameter correlation function has a universal equal-time scaling form

$$C(12)\equiv\langle\vec{\psi}(1)\cdot\vec{\psi}(2)\rangle=\psi_0^2\mathcal{F}(x), \tag{6}$$

where ψ_0 is the magnitude $\psi=|\vec{\psi}|$ of the order parameter in the ordered phase. Here we use the shorthand notation where 1 denotes $(\mathbf{r}_1,t_1)$. The scaled length x is defined as $\vec{x}=(\vec{r}_1-\vec{r}_2)/L(t)$, where $L(t)\sim t^{1/2}$ for the nonconserved models considered here.

In previous work on the order parameter scaling function it was important to make a mapping of the order parameter $\vec{\psi}$ onto an auxiliary field $\vec{m}$ with the key requirement that *away* from defect cores

$$\vec{\psi}=\psi_0\hat{m} \tag{7}$$

for the lowest-energy defects having unit topological charge. Physically, one expects that *near* the defect cores

$$\vec{\psi}=a\vec{m}+b(\vec{m})^2\vec{m}+\cdots \tag{8}$$

for charge ±1 defects where a and b are constants. In the theory for the order-parameter correlations it is property (7) that is important. In the theory of defect motion, as presented here, it is property (8) that is important. In this paper only property (8) enters into the analysis since we always work near the defect cores. To complete the definition of the model one must specify the form of the probability distribution for the auxiliary field $\vec{m}$. The simplest choice is a Gaussian probability distribution for $\vec{m}$ with

$$\langle m_\nu(1)m_{\nu'}(2)\rangle=\delta_{\nu\nu'}C_0(12). \tag{9}$$

The system is assumed to be statistically isotropic and homogeneous so $C_0(12)$ is invariant under interchange of its spatial indices. In the scaling regime at equal times $(t_1=t_2=t)$ we introduce the auxiliary field autocorrelation function mentioned in the Introduction,

$$f(x)=C_0(\vec{r}_1t,\vec{r}_2t)/S_0(t), \tag{10}$$

and $S_0(t)=C_0(11)$ grows as $L^2(t)$ with time after the quench.

III. TOPOLOGICAL DEFECTS

A. Densities

It has been emphasized in Ref. [7] that the signed or charged point $(n=d)$ defect density can be written in the form

$$\rho(\mathbf{R},t)=\delta(\vec{\psi}(\mathbf{R},t))\mathcal{D}(\vec{R},t), \tag{11}$$

where the Jacobian obtained with the change of variables from the set of vortex positions to the zeros of the field $\vec{\psi}$ is defined by

$$\mathcal{D}(\mathbf{R},t)=\frac{1}{n!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\nabla_{\mu_1}\psi_{\nu_1}\times\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}, \tag{12}$$

where $\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}$ is the n-dimensional fully antisymmetric tensor and summation over repeated indices is implied. The key point is that the zeros of the order parameter $\vec{\psi}$ locate the positions of the vortices. The unsigned density $n(\mathbf{R},t)$ is given by

$$n(\mathbf{R},t)=\delta(\vec{\psi}(\mathbf{R},t))|\mathcal{D}(\vec{R},t)|. \tag{13}$$

The charged vortex correlation function is given by

$$C_{\rho\rho}(\mathbf{R},t)=\langle\rho(\mathbf{R},t)\rho(0,t)\rangle, \tag{14}$$

while the unsigned vortex correlation function is given by

$$C_{nn}(\mathbf{R},t)=\langle n(\mathbf{R},t)n(\mathbf{0},t)\rangle. \tag{15}$$

It is shown in Ref. [5] that the vortex-vortex and vortex-antivortex correlation functions can be expressed in terms of $C_{\rho\rho}$ and C_{nn}. $C_{\rho\rho}$ was evaluated in Ref. [7] using the Gaussian closure approximation. As shown in Ref. [5], the evaluation of C_{nn} in this same approximation is technically much more difficult than the calculation of $C_{\rho\rho}$ because of the absolute value sign in the definition of the unsigned defect density n.

B. Conservation of topological charge

It was shown in Ref. [6] that the charged vortex density satisfies the continuity equation

$$\frac{\partial\rho}{\partial t}=\nabla_\beta[\delta(\vec{\psi})J_\beta^{(K)}], \tag{16}$$

where

$$J_\alpha^{(K)}=\frac{1}{(n-1)!}\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}K_{\nu_1}\times\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}. \tag{17}$$

A key point here is that $J_\beta^{(K)}$ is multiplied by the vortex locating δ function. This means that one can replace $\vec{K}$ in $\vec{J}^{(K)}$ by the part of $\vec{K}$ that does not vanish as $\vec{\psi}\rightarrow 0$. Thus, in the case of a nonconserved order parameter one can replace $J_\beta^{(K)}$ in the continuity equation by

$$J_\beta^{(2)}=\frac{\Gamma c}{(n-1)!}\epsilon_{\beta,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\nabla^2\psi_{\nu_1}\times\nabla_{\mu_2}\psi_{\nu_2}\cdots\nabla_{\mu_n}\psi_{\nu_n}. \tag{18}$$

Because of the standard form of the continuity equation (16), it is clear that one can identify the vortex velocity field as

$$v_\alpha=-\frac{J_\alpha^{(2)}}{\mathcal{D}}. \tag{19}$$

This form for the velocity field is used inside expressions multiplied by the vortex locating δ function.

C. Use of topological charge conservation to determine the auxiliary field correlation function

In previous work [17,18] a rather successful scheme has been developed for evaluating the order-parameter correlation function $\mathcal{F}(x)$ and in turn the auxiliary field correlation function $f(x)$. As indicated in the Introduction, this leads to the problem that the auxiliary field correlation function is rendered nonanalytic as a function of x for small x. Mazenko and Wickham [14] have shown recently that this problem can be addressed in a different way. Rather than using the order-parameter equation of motion to determine order-parameter correlation function they used the continuity equation for topological charge to determine the auxiliary field correlation function. As in the rest of this paper, we use the fact that in quantities such as ρ and $\vec{v}$, we can replace $\vec{\psi}\rightarrow\vec{m}$ everywhere. Then we can determine $f(x)$ by satisfying

$$\frac{\partial}{\partial t}\langle\rho(1)\rho(2)\rangle=\nabla_{(1)}^{\beta}\langle\delta(\vec{\psi}(1))J_\beta^{(2)}(1)\rho(2)\rangle+\nabla_{(2)}^{\beta}\langle\rho(1)\delta(\vec{\psi}(2))J_\beta^{(2)}(2)\rangle \tag{20}$$

under the assumption that $\vec{m}$ is a Gaussian field. The calculation of the left-hand side of Eq. (20) amounts to the evaluation of $C_{\rho\rho}$. This calculation was carried out in Ref. [7] and is straightforward since $C_{\rho\rho}$ factorizes into a product of Gaussian averages that can be evaluated using standard methods. The calculation of the average over $J_\beta^{(2)}$ can be organized in a similar fashion. In the scaling regime, after an impressive set of cancellations, one finds the rather simple result that

$$-\mu x f'=\nabla^2 f+n\frac{S^{(2)}}{\sigma}f, \tag{21}$$

where

$$S^{(2)}=\frac{1}{n^2}\langle(\nabla\vec{m})^2\rangle, \tag{22}$$

$$S_0=\sigma L^2, \tag{23}$$

and we introduce the constant

$$\mu=\frac{L\dot{L}}{2\Gamma c}. \tag{24}$$

This equation for f is linear and has the simple solution of the OJK form

$$f=e^{-(\mu/2)x^2}, \tag{25}$$

with the conditions

$$n\frac{S^{(2)}}{\sigma}=(-\nabla^2 f)|_{x=0}=n\mu. \tag{26}$$

For simplicity we set $\mu=1$, which amounts to choosing $L(t)=2\sqrt{\Gamma ct}$ and results in the result for f given by Eq. (3).

D. Vortex velocities

As an important application of the result (19) for the vortex velocity field $\vec{v}$ consider the velocity probability distribution function defined by

$$n_0 P(\vec{v}_1)\equiv\langle n\,\delta(\vec{v}_1-\vec{v})\rangle, \tag{27}$$

where $\vec{v}_1$ is a reference velocity, $\vec{v}$ is given by Eq. (19), n is the unsigned defect density, and $n_0=\langle n\rangle$. $P[\vec{v}_1]$ was found in Ref. [6] to be given by

$$P(\vec{v}_1)=\frac{\Gamma\left(\frac{n}{2}+1\right)}{(\pi\overline{v^2})^{n/2}}\frac{1}{[1+(\vec{v}_1)^2/\overline{v^2}]^{(n+2)/2}}, \tag{28}$$

where the characteristic velocity $\overline{v}$ is defined by

$$\overline{v^2}=(\Gamma c)^2\frac{\overline{S_4}}{S^{(2)}}, \tag{29}$$

where $S^{(2)}$ is given by Eq. (22) and

$$\overline{S_4}=\frac{1}{n}\langle(\nabla^2\vec{m})^2\rangle-\frac{(nS^{(2)})^2}{S_0}. \tag{30}$$

Using the OJK form for $f(x)$, we obtain $S^{(2)}=\sigma, \overline{S_4}=d\sigma/\Gamma ct$, and $\overline{v^2}=d\Gamma c/t$.

IV. CALCULATION OF THE TWO-VORTEX VELOCITY PROBABILITY DISTRIBUTION

A. General development

The main quantity of interest in this paper is the two-velocity correlation function defined by

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\langle n(1)\,\delta(\vec{v}_1-\vec{v}(1))n(2)\,\delta(\vec{v}_2-\vec{v}(2))\rangle, \tag{31}$$

where $\vec{v}_1$ and $\vec{v}_2$ are external labels while the $\vec{v}(i)$, for $i=1,2$, is expressed in terms of the order-parameter field $\psi(i)$ via Eq. (19). $C_{nn}P[\vec{v}_1,\vec{v}_2]$ is normalized such that the integrals over $\vec{v}_1$ and $\vec{v}_2$ give the unsigned defect density correlation function C_{nn}, which was determined previously in Ref. [5].

The first step in the evaluation of $P[\vec{v}_1,\vec{v}_2]$ is to notice that we can replace $\vec{\psi}$ by $\vec{m}$ in the expressions for the unsigned vortex density and the velocity. Next we need to show that it can be expressed in terms of an average over a reduced probability distribution. In the Appendix we introduce the fields

$$W_i[\xi,\vec{b}]\equiv\delta(\vec{m}(i))\,\delta(\vec{b}(i)-\nabla_i^2\vec{m}(i))\times\prod_{\mu,\nu=1}^{n}\delta(\xi_\mu^\nu(i)-\nabla_\mu^{(i)}m_\nu(i)), \tag{32}$$

which have the normalization

$$\int d^n b(i)\prod_{\mu,\nu=1}^{n}d\xi_\mu^\nu(i)W_i[\xi,\vec{b}]=\delta(\vec{m}(i)). \tag{33}$$

Using this result, we can insert the factors of W_1W_2 into the expression for $P[\vec{v}_1,\vec{v}_2]$ and use the properties of the δ function to replace all gradients and Laplacians of $\vec{m}$ with the associated values constrained by the multiplying δ function to obtain

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\int\prod_{i=1}^{2}\left[d^n b(i)\prod_{\mu,\nu=1}^{n}d\xi_\mu^\nu(i)|\mathcal{D}(\xi(i))|\times\delta(\vec{v}_i-\vec{v}(\xi(i),\vec{b}(i)))\right]G_2(\xi,\vec{b}),$$

where

$$G_2(\xi,\vec{b})\equiv\langle W_1[\xi,\vec{b}]W_2[\xi,\vec{b}]\rangle, \tag{34}$$

$$\mathcal{D}(\xi)=\frac{1}{n!}\epsilon_{\mu_1,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\xi_{\mu_1}^{\nu_1}\xi_{\mu_2}^{\nu_2}\cdots\xi_{\mu_n}^{\nu_n}, \tag{35}$$

$$v_\alpha(\xi(i),\vec{b}(i))=-\frac{J_\alpha^{(2)}(\xi(i),\vec{b}(i))}{\mathcal{D}(\xi(i))}, \tag{36}$$

with

$$J_\alpha^{(2)}(\xi(i),\vec{b}(i))=\frac{\Gamma c}{(n-1)!}\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}\times b_{\nu_1}(i)\xi_{\mu_2}^{\nu_2}(i)\cdots\xi_{\mu_n}^{\nu_n}(i). \tag{37}$$

The Gaussian average giving $G_2(\xi,\vec{b})$ is worked out explicitly in the Appendix. In the course of this calculation it is required that one make a change of variables from $\xi_\mu^\nu(i)$ to a new set $t_\mu^\nu(i)$ given by

$$\xi_\mu^\nu(i)=\hat{R}_\mu^\beta t_\beta^\nu(i) \tag{38}$$

and $\hat{R}_\mu^\beta$ is an orthonormal matrix with the additional property that $\hat{R}_\mu^{(1)}=\hat{R}_\mu$, where $\hat{R}_\mu$ is the unit vector pointing from vortex 2 to vortex 1. Since $\det(\hat{R})=1$ the change of variables from ξ to t is simple,

$$\prod_{\nu,\mu,j}d\xi_\mu^\nu(j)=\prod_{\nu,\mu,j}dt_\mu^\nu(j) \tag{39}$$

and

$$\mathcal{D}(\xi(j))=\mathcal{D}(t(j)). \tag{40}$$

The one place in this change of variables where one must show care is for the current $\vec{J}^{(2)}$. We have

$$J_\alpha^{(2)}(\xi(i),\vec{b}(i))=\frac{\Gamma c}{(n-1)!}\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}b_{\nu_1}(i)\hat{R}_{\mu_2}^{\beta_2}t_{\beta_2}^{\nu_2}(i)\cdots\hat{R}_{\mu_n}^{\beta_n}t_{\beta_n}^{\nu_n}(i)$$

$$=\frac{\Gamma c}{(n-1)!}\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\hat{R}_{\mu_2}^{\beta_2},\ldots,\hat{R}_{\mu_n}^{\beta_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}b_{\nu_1}(i)t_{\beta_2}^{\nu_2}(i),\ldots,t_{\beta_n}^{\nu_n}(i). \tag{41}$$

Clearly, if we multiply this expression by $\hat{R}_\alpha^{\beta_1}$ and sum over α we obtain

$$\hat{R}_\alpha^{\beta_1}J_\alpha^{(2)}(\xi(i),\vec{b}(i))=\frac{\Gamma c}{(n-1)!}\hat{R}_\alpha^{\beta_1}\epsilon_{\alpha,\mu_2,\ldots,\mu_n}\hat{R}_{\mu_2}^{\beta_2},\ldots,\hat{R}_{\mu_n}^{\beta_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}b_{\nu_1}(i)t_{\beta_2}^{\nu_2}(i),\ldots,t_{\beta_n}^{\nu_n}(i)$$

$$=\frac{\Gamma c}{(n-1)!}(\det\hat{R})\epsilon_{\beta_1,\beta_2,\ldots,\beta_n}\epsilon_{\nu_1,\nu_2,\ldots,\nu_n}b_{\nu_1}(i)t_{\beta_2}^{\nu_2}(i),\ldots,t_{\beta_n}^{\nu_n}(i)=J_\alpha^{(2)}(\mathbf{t}(i),\vec{b}(i)). \tag{42}$$

Multiplying by $\hat{R}_\mu^{\beta_1}$, summing over β_1, and using the orthonormality of the matrix $\hat{R}$ gives

$$J_\mu^{(2)}(\xi(i),\vec{b}(i))=\hat{R}_\mu^{\beta_1}J_{\beta_1}^{(2)}(t(i),\vec{b}(i)). \tag{43}$$

Because of the rotational invariance of the d-dimensional δ function we have

$$\delta(\vec{v}_i-\vec{v}(\xi(i),\vec{b}(i)))=\delta(\vec{u}(i)-\vec{v}(t(i),\vec{b}(i))), \tag{44}$$

where

$$u(i)_\mu=\hat{R}_\beta^\mu v_{i,\beta}. \tag{45}$$

Thus the $\mu=1$ component of u_μ is the longitudinal component along $\hat{R}$. We then have that

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\int\prod_{i=1}^{2}\left[d^nb(i)\prod_{\mu,\nu=1}^{n}dt_\mu^\nu(i)|\mathcal{D}(t(i))|\times\delta(\vec{u}(i)-\vec{v}(t(i),\vec{b}(i)))\right]G_2(t,\vec{b}). \tag{46}$$

The next step in the analysis is to perform the integration over the $\vec{b}$ variables. Toward this end we use the representation

$$\delta(\vec{u}(i)-\vec{v}(t(i),\vec{b}(i)))=\int\frac{d^nz(i)}{(2\pi)^n}e^{-i\vec{u}(i)\cdot\vec{z}(i)}e^{i\vec{v}(t(i),\vec{b}(i))\cdot\vec{z}(i)} \tag{47}$$

and we make explicit the $\vec{b}(i)$ dependence by writing

$$\vec{v}(t(i),\vec{b}(i))\cdot\vec{z}(i)\equiv a_\nu(i)b_\nu(i), \tag{48}$$

where

$$a_\nu(i)=-\frac{\Gamma c}{(n-1)!}\frac{1}{\mathcal{D}(t(i))}z_\alpha(i)\epsilon_{\alpha\mu_2\cdots\mu_n}\epsilon_{\nu\nu_2\cdots\nu_n}\times t_{\mu_2}^{\nu_2}(i)\cdots t_{\mu_n}^{\nu_n}(i)=z_\alpha(i)N_{\nu\alpha}(i) \tag{49}$$

and

$$N_{\nu\alpha}(i)=-\frac{\Gamma c}{(n-1)!}\frac{1}{\mathcal{D}(t(i))}\epsilon_{\alpha\mu_2\cdots\mu_n}\epsilon_{\nu\nu_2\cdots\nu_n}\times t_{\mu_2}^{\nu_2}(i)\cdots t_{\mu_n}^{\nu_n}(i). \tag{50}$$

Next one must make the $\vec{b}$ dependence of $G_2(t,\vec{b})$ explicit. We have from the Appendix that

$$G_2(\xi,\vec{b})=G_T(t_T)G_L(\vec{b},\vec{t}_L). \tag{51}$$

The transverse part of G_2 does not depend on $\vec{b}(i)$, while the longitudinal contribution can be written as

$$G_L(\vec{b},\vec{t}_L)=\frac{1}{(2\pi)^{3n}}\frac{1}{(\det M)^{n/2}}\times\exp\left(-\frac{1}{2}\sum_{\alpha,\beta=1}^{6}\vec{h}_\alpha\cdot\vec{h}_\beta(M^{-1})_{\alpha\beta}\right), \tag{52}$$

where the matrix M is discussed in detail in the Appendix and the $\vec{h}_\alpha$ are defined by Eqs. (A47)–(A51). Using the explicit expressions for the $\vec{h}_\alpha$ we can write

$$\sum_{\alpha,\beta=1}^{6}\vec{h}_\alpha\cdot\vec{h}_\beta(M^{-1})_{\alpha\beta}=S_b\sum_i\vec{b}(i)^2+2C_b\vec{b}(1)\cdot\vec{b}(2)+2\sum_i\vec{b}(i)\cdot\vec{S}(i)+S_L^0\sum_i\vec{t}_L^2(i)+2C_L^0\vec{t}_L(1)\cdot\vec{t}_L(2), \tag{53}$$

where we have defined

$$S_b=(M^{-1})_{33}=(M^{-1})_{44}, \tag{54}$$

$$C_b=(M^{-1})_{34}=(M^{-1})_{43}, \tag{55}$$

$$\vec{S}(1)=(M^{-1})_{35}\vec{t}_L(1)+(M^{-1})_{36}\vec{t}_L(2), \tag{56}$$

$$\vec{S}(2)=-(M^{-1})_{36}\vec{t}_L(1)-(M^{-1})_{35}\vec{t}_L(2), \tag{57}$$

$$S_L^0=(M^{-1})_{55}=(M^{-1})_{66}, \tag{58}$$

$$C_L^0=(M^{-1})_{56}=(M^{-1})_{65}. \tag{59}$$

The matrix inverse M^{-1} is also discussed in detail in the Appendix. It is convenient to define

$$G_L^0(\vec{b},\vec{t}_L)=\frac{1}{(2\pi)^{3n}}\frac{1}{(\det M)^{n/2}}\exp\left[-\frac{1}{2}\left(S_L^0\sum_i \vec{t}_L^{\,2}(i)+2C_L^0\vec{t}_L(1)\cdot\vec{t}_L(2)\right)\right] \tag{60}$$

and the b dependence is explicit when we write

$$G_2(\xi,\vec{b})=G_T(t_T)G_L^0(\vec{t}_L)\exp\left[-\frac{1}{2}\left(S_b\sum_i \vec{b}(i)^2+2C_b\vec{b}(1)\cdot\vec{b}(2)+2\sum_i \vec{b}(i)\cdot\vec{S}(i)\right)\right]. \tag{61}$$

Putting this together we have

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\int\prod_{i=1}^{2}\left[\prod_{\mu,\nu=1}^{n}dt_\mu^\nu(i)\frac{d^nz(i)}{(2\pi)^n}|\mathcal{D}(t(i))|\right]\times G_T(t_T)G_L^0(\vec{t}_L)\exp\left(-i\sum_i \vec{u}(i)\cdot\vec{z}(i)\right)J_0, \tag{62}$$

where the $\vec{b}(i)$ integrations are isolated in

$$J_0=\int\prod_i d^nb(i)\exp\left(i\sum_i \vec{b}(i)\cdot[\vec{a}(i)+i\vec{S}(i)]\right)\times\exp\left[-\frac{1}{2}\left(S_b\sum_i \vec{b}(i)^2+2C_b\vec{b}(1)\cdot\vec{b}(2)\right)\right]. \tag{63}$$

This integration is of the standard Gaussian form. If we define

$$\vec{A}(i)=\vec{a}(i)+i\vec{S}(i), \tag{64}$$

then we have the result

$$J_0=(2\pi)^n\left(\frac{\gamma_b}{S_b}\right)^n\exp\left(-\frac{1}{2}\frac{\gamma_b^2}{S_b}\mathcal{Q}\right), \tag{65}$$

where

$$\mathcal{Q}=\sum_{i=1}^{2}\vec{A}^2(i)-2f_b\vec{A}(1)\cdot\vec{A}(2) \tag{66}$$

and

$$\gamma_b=(1-f_b^2)^{-1/2}, \tag{67}$$

$$f_b=C_b/S_b. \tag{68}$$

The next step is to do the $\vec{z}(i)$ integrations. We can highlight the z dependence if we remember that

$$a_\nu(i)=z_\alpha(i)N_{\nu\alpha}(i). \tag{69}$$

It is then a matter of straightforward algebra to show that

$$\mathcal{Q}=-\sum_i \vec{S}^2(i)+2f_b\vec{S}(1)\cdot\vec{S}(2)+\frac{S_b}{\gamma_b^2}\sum_{\alpha,\beta}\sum_{i,j}z_\alpha(i)E_{\alpha\beta}(ij)z_\beta(j), \tag{70}$$

where

$$E_{\alpha\beta}(ij)=\frac{\gamma_b^2}{S_b}\Omega_{\alpha\beta}(ij)[\delta_{ij}-f_b\delta_{j,i+1}] \tag{71}$$

and

$$\Omega_{\alpha\beta}(ij)=\sum_\nu N_{\nu\alpha}(i)N_{\nu\beta}(j). \tag{72}$$

Here we have introduced the convenient notation that the index i is periodic, so that if $i=2$, then $i+1=1$. Putting these results together we have

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\int\prod_{i=1}^{2}\left[\prod_{\mu,\nu=1}^{n}dt_\mu^\nu(i)|\mathcal{D}(t(i))|\right]\times G_T(t_T)G_L^0(\vec{t}_L)(2\pi)^n\left(\frac{\gamma_b}{S_b}\right)^n\times\exp\left[\frac{1}{2}\frac{\gamma_b^2}{S_b}\left(\sum_i \vec{S}^2(i)-2f_b\vec{S}(1)\cdot\vec{S}(2)\right)\right]J_1, \tag{73}$$

where the z integration is given explicitly by

$$J_1=\int\prod_{i=1}^{2}\left[\frac{d^nz(i)}{(2\pi)^n}\right]\exp\left(-i\sum_i \vec{U}(i)\cdot\vec{z}(i)\right)\times\exp\left(-\frac{1}{2}\sum_{\alpha,\beta}\sum_{i,j}z_\alpha(i)E_{\alpha\beta}(ij)z_\beta(j)\right), \tag{74}$$

where

$$\vec{U}_i=\vec{u}(i)+\vec{d}(i) \tag{75}$$

and

$$d_\alpha(i)=\frac{\gamma_b^2}{S_b}\sum_{\nu=1}^{n}N_{\nu\alpha}(i)[S_\nu(i)-f_bS_\nu(i+1)]. \tag{76}$$

J_1 is again of the standard form for a Gaussian integral, so

$$J_1=\frac{1}{(2\pi)^{n/2}}\frac{1}{\sqrt{\det E}}\exp\left[-\frac{1}{2}U_\alpha(i)(E^{-1})_{\alpha\beta}(ij)U_\beta(j)\right], \tag{77}$$

where, again, we need the determinant and the inverse of a matrix, in this case E. Let us look at the inverse first. If we note the important result (used in Ref. [6])

$$N_{\sigma\alpha}(i)t_{\alpha}^{\nu}(i)=-\Gamma c\,\delta_{\sigma\nu}, \tag{78}$$

where we do not sum on i, then

$$\Omega_{\alpha\beta}(ij)t_{\beta}^{\sigma}(j)=-\Gamma c N_{\alpha\sigma}, \tag{79}$$

where we do not sum on j. These identities suggest that we try a solution for E^{-1} of the form

$$(E^{-1})_{\alpha\beta}(ij)=\sum_{\nu} t_{\alpha}^{\nu}(i)e_{ij}t_{\beta}^{\nu}(j), \tag{80}$$

with e_{ij} to be determined. Inserting this ansatz into the equation defining the inverse we easily find that

$$e_{ij}=\frac{1}{(\Gamma c)^2}[S_b\delta_{ij}+C_b\delta_{j,i+1}]. \tag{81}$$

In computing $\det E$ we use the fact that

$$\det E=\frac{1}{\det(E^{-1})} \tag{82}$$

and that E^{-1} can be written as the matrix product

$$(E^{-1})_{\alpha\beta}(ij)=\sum_{\nu,\nu',k,\ell} t_{\alpha}^{\nu}(i)\,\delta_{ik}e_{k\ell}\delta_{\nu\nu'}\delta_{\ell j}t_{\beta}^{\nu'}(j), \tag{83}$$

so that

$$\begin{aligned}\det E^{-1}&=\det t_{\alpha}^{\nu}(i)\,\delta_{ik}\det(e_{k\ell})\,\delta_{\nu\nu'}\det\delta_{\ell j}t_{\beta}^{\nu'}(j)\\&=\mathcal{D}(t(1))\mathcal{D}(t(2))(\det e)^n\mathcal{D}(t(1))\mathcal{D}(t(2))\end{aligned} \tag{84}$$

and

$$\det e=\frac{1}{(\Gamma c)^4}(S_b^2-C_b^2)=\frac{1}{(\Gamma c)^4}\frac{S_b^2}{\gamma_b^2}. \tag{85}$$

Pulling all of this together leads to the result

$$\begin{aligned}C_{nn}P[\vec{v}_1,\vec{v}_2]=&\frac{1}{(\Gamma c)^{2n}}\int\left[\prod_{i=1}^{2}\prod_{\mu,\nu=1}^{n}dt_{\mu}^{\nu}(i)\mathcal{D}^2(t(i))\right]\\&\times G_T(t_T)G_L^0(\vec{t}_L)\exp\left[\frac{1}{2}\frac{\gamma_b^2}{S_b}\left(\sum_i \vec{S}^2(i)\right.\right.\\&\left.\left.-2f_b\vec{S}(1)\cdot\vec{S}(2)\right)\right]\exp\left[-\frac{1}{2}\sum_{i,j,\alpha,\beta}U_{\alpha}(i)\right.\\&\left.\times(E^{-1})_{\alpha\beta}(ij)U_{\beta}(j)\right]\end{aligned} \tag{86}$$

and the important point is that one does not have an absolute value sign left in the Jacobian factors. Turn next to the argument of the exponential in the last line of Eq. (86). After a substantial amount of algebra we find

$$\begin{aligned}&-\frac{\gamma_b^2}{S_b}\left[\sum_i \vec{S}^2(i)-2f_b\vec{S}(1)\cdot\vec{S}(2)\right]+\sum_{i,j,\alpha,\beta}U_{\alpha}(i)\\&\times(E^{-1})_{\alpha\beta}(ij)U_{\beta}(j)\\&=\sum_{i,j}\vec{\mathcal{V}}(i)\cdot\vec{\mathcal{V}}(j)e_{ij}-\frac{2}{\Gamma c}\sum_i\vec{\mathcal{V}}(i)\cdot\vec{S}(i),\end{aligned} \tag{87}$$

where

$$\mathcal{V}_{\nu}(i)=\sum_{\alpha}u_{\alpha}(i)t_{\alpha}^{\nu}(i). \tag{88}$$

Putting this together leads to

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\frac{1}{(2\pi)^{3n}}\frac{1}{(\Gamma c)^{2n}}\frac{1}{(\det M)^{n/2}}\left(\frac{\gamma_T}{2\pi S_T}\right)^{n(n-1)}J_2, \tag{89}$$

where the final integration is over the matrices $t_{\alpha}^{\beta}(i)$:

$$J_2=\int\left[\prod_{i=1}^{2}\prod_{\mu,\nu=1}^{n}dt_{\mu}^{\nu}(i)\mathcal{D}^2(t(i))\right]e^{-(1/2)A(t)}, \tag{90}$$

where

$$\begin{aligned}A(t)=&\frac{\gamma_T^2}{S_T}\sum_{\mu=2}^{n}\sum_{\nu=1}^{n}\left[\sum_{i=1}^{2}[t_{\mu}^{\nu}(i)]^2-2f_Tt_{\mu}^{\nu}(1)t_{\mu}^{\nu}(2)\right]\\&+S_L^0\sum_i\vec{t}_L^2(i)+2C_L^0\vec{t}_L(1)\cdot\vec{t}_L(2)+\sum_{i,j}\vec{\mathcal{V}}(i)\cdot\vec{\mathcal{V}}(j)e_{ij}\\&-\frac{2}{\Gamma c}\sum_i\vec{\mathcal{V}}(i)\cdot\vec{S}(i).\end{aligned} \tag{91}$$

$A(t)$ is clearly a quadratic form in the matrix $t_{\mu}^{\nu}(i)$. The quantities S_T, f_T, and γ_T govern the transverse modes and are defined by Eqs. (A56)–(A59) in the Appendix. After sufficient rearrangement $A(t)$ can be written in the final form

$$A(t)=\sum_{i,j,\alpha,\beta,\nu}t_{\alpha}^{\nu}(i)W_{\alpha\beta}(ij)t_{\beta}^{\nu}(j), \tag{92}$$

where the matrix W plays a central role in the theory and is given by the manifestly symmetric form

$$\begin{aligned}W_{\alpha\beta}(ij)=&\,\delta_{\alpha\beta}d_{ij}^{\alpha}+u_{\alpha}(i)\Omega_{ij}^{(1)}u_{\beta}(j)+u_{\alpha}(i)\Omega_{ij}^{(2)}L_{\beta}(j)\\&+L_{\alpha}(i)\Omega_{ij}^{(2)}u_{\beta}(j),\end{aligned} \tag{93}$$

where

$$d_{ij}^{\alpha}=\delta_{ij}d_{\alpha}+\delta_{i+1,j}d_{\alpha}^{c}, \tag{94}$$

$$d_{\alpha}=\delta_{\alpha,L}S_L^0+(1-\delta_{\alpha,L})\frac{\gamma_T^2}{S_T}, \tag{95}$$

$$d_{\alpha}^{c}=\delta_{\alpha,L}C_L^0+(1-\delta_{\alpha,L})f_T\frac{\gamma_T^2}{S_T}, \tag{96}$$

$$\Omega_{ij}^{(1)}=\delta_{ij}\frac{S_b}{(\Gamma c)^2}+\delta_{i+1,j}\frac{C_b}{(\Gamma c)^2}, \tag{97}$$

$$\Omega_{ij}^{(2)}=-\delta_{ij}\frac{(M^{-1})_{35}}{\Gamma c}+\delta_{i+1,j}\frac{(M^{-1})_{36}}{\Gamma c}, \tag{98}$$

and

$$L_\alpha(i)=\delta_{\alpha,L}\eta_i, \tag{99}$$

where

$$\eta_1=-\eta_2=1. \tag{100}$$

The final integration over the matrices $t_\alpha^\nu(i)$ is not of the standard form evaluated so far, but instead there is the polynomial $\mathcal{D}^2(1)\mathcal{D}^2(2)$ multiplying the Gaussian in the integrand. It is technically important that there are no absolute value signs left in this expression and the integral can be evaluated by introducing a field $g_\alpha^\nu(i)$ that couples to $t_\alpha^\nu(i)$ via

$$-\frac{1}{2}A(t)\rightarrow-\frac{1}{2}A(t)+\sum_{\alpha,\nu,i}g_\alpha^\nu(i)t_\alpha^\nu(i). \tag{101}$$

If we consider

$$J_2(g)=\int\left[\prod_{i=1}^{2}\prod_{\mu,\nu=1}^{n}dt_\mu^\nu(i)\mathcal{D}^2(t(i))\right]\times\exp\left(-\frac{1}{2}A(t)e\sum_{\alpha,\nu,i}g_\alpha^\nu(i)t_\alpha^\nu(i)\right), \tag{102}$$

then any polynomial can be generated by taking derivatives with respect to g. Using the explicit expressions for the $\mathcal{D}^2(t(i))$ we have

$$J_2(g)=\sum_{\mu_1\cdots\mu_n=1}^{n}\epsilon_{\mu_1\cdots\mu_n}\frac{\partial}{\partial g_{\mu_1}^1(1)}\cdots\frac{\partial}{\partial g_{\mu_n}^n(1)}\times\sum_{\mu_1'\cdots\mu_n'=1}^{n}\epsilon_{\mu_1'\cdots\mu_n'}\frac{\partial}{\partial g_{\mu_1'}^1(1)}\cdots\frac{\partial}{\partial g_{\mu_n'}^n(1)}\times\sum_{\nu_1\cdots\nu_n=1}^{n}\epsilon_{\nu_1\cdots\nu_n}\frac{\partial}{\partial g_{\nu_1}^1(2)}\cdots\frac{\partial}{\partial g_{\nu_n}^n(2)}\times\sum_{\nu_1'\cdots\nu_n'=1}^{n}\epsilon_{\nu_1'\cdots\nu_n'}\frac{\partial}{\partial g_{\nu_1'}^1(2)}\cdots\frac{\partial}{\partial g_{\nu_n'}^n(2)}J_3(g), \tag{103}$$

where

$$J_3(g)=\int\left[\prod_{i=1}^{2}\prod_{\mu,\nu=1}^{n}dt_\mu^\nu(i)\right]\exp\left(-\frac{1}{2}A(t)e\sum_{\alpha,\nu,i}g_\alpha^\nu(i)t_\alpha^\nu(i)\right). \tag{104}$$

$J_3(g)$ is now of the standard form and we have

$$J_3(g)=\frac{(2\pi)^{n^2}}{(\det W)^{n/2}}\exp\left[\frac{1}{2}\sum_{i,j,\alpha,\beta,\nu}g_\alpha^\nu(i)\Lambda_{\alpha\beta}(ij)g_\beta^\nu(j)\right], \tag{105}$$

where $\Lambda_{\alpha\beta}(ij)$ is the matrix inverse of W. It is then straightforward to take the derivatives with respect to g and then set g to zero to obtain

$$J_2=\frac{(2\pi)^{n^2}}{(\det W)^{n/2}}J_4, \tag{106}$$

where

$$J_4=\sum_{\mu_1\cdots\mu_n=1}^{n}\epsilon_{\mu_1\cdots\mu_n}\sum_{\mu_1'\cdots\mu_n'=1}^{n}\epsilon_{\mu_1'\cdots\mu_n'}\sum_{\nu_1\cdots\nu_n=1}^{n}\epsilon_{\nu_1\cdots\nu_n}\sum_{\nu_1'\cdots\nu_n'=1}^{n}\epsilon_{\nu_1'\cdots\nu_n'}\prod_{\sigma=1}^{n}[\Lambda_{\mu_\sigma\mu_\sigma'}(11)\Lambda_{\nu_\sigma'\nu_\sigma}(22)+\Lambda_{\mu_\sigma\nu_\sigma'}(12)\Lambda_{\mu_\sigma'\nu_\sigma}(12)+\Lambda_{\mu_\sigma\nu_\sigma}(12)\Lambda_{\mu_\sigma'\nu_\sigma'}(12)], \tag{107}$$

and the final result is

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\frac{1}{(2\pi)^{3n}}\frac{1}{(\Gamma c)^{2n}}\frac{1}{(\det M)^{n/2}}\left(\frac{\gamma_T}{2\pi S_T}\right)^{n(n-1)}\times\frac{(2\pi)^{n^2}}{(\det W)^{n/2}}J_4. \tag{108}$$

Finally, all of the integrals have been evaluated. What is left is to evaluate the determinant and matrix inverse of W.

B. Large-*x* limit

As a check on the preceding analysis it is useful to work out the large-scaled distance limit where we expect the probability distribution to factorize into the product for each of the tagged vortices. In this limit, using the results from Table I, we find that the matrices entering W are in a diagonal form

$$d_{ij}^\alpha=D\delta_{ij}, \tag{109}$$

where

$$D=\frac{1}{\sigma}, \tag{110}$$

$$\Omega_{ij}^{(1)}=\frac{L^2}{2d\sigma(\Gamma c)^2}\delta_{ij}, \tag{111}$$

TABLE I. Small-x (left) and large-x values for various quantities (far left) defined in the text.

$\kappa_E^{(1)}$	$2x$	0
$\kappa_E^{(2)}$	16	$2n$
$\kappa_E^{(0)}$	x^2	1
$\kappa_O^{(0)}$	$\frac{x^4}{12}$	0
$\kappa_O^{(1)}$	$-\frac{x^3}{2}$	$-2n$
$\kappa_O^{(2)}$	$-2x^4$	1
D_E	$16\sigma^3x^2$	$4n\sigma^3$
D_O	$\sigma^3\frac{x^8}{48}$	$n\sigma^3$
det$M$	$\frac{\sigma^6x^{10}}{3}$	$4n^2\sigma^6$

and

$$\Omega_{ij}^{(2)}=0. \tag{112}$$

The matrix W can then be written in the partially diagonal form

$$W_{\alpha\beta}(ij)=\delta_{ij}[\delta_{\alpha\beta}D+\overline{u}_\alpha(i)\overline{u}_\beta(i)], \tag{113}$$

where

$$\overline{u}_\alpha(i)=\frac{Lu_\alpha(i)}{\Gamma c\sqrt{2n\sigma}}. \tag{114}$$

Notice that there is no longer a difference between the longitudinal and transverse directions as expected. The inverse matrix Λ then satisfies the equation

$$D\Lambda_{\alpha\beta}(ij)+\overline{u}_\alpha(i)\sum_\mu \overline{u}_\mu(i)\Lambda_{\mu\beta}(ij)=\delta_{\alpha\beta}\delta_{ij}. \tag{115}$$

This equation is in the form of a trap that can first be solved to obtain

$$\sum_\mu \overline{u}_\mu(i)\Lambda_{\mu\beta}(ij)=\delta_{ij}\frac{\overline{u}_\beta(i)/D}{1+\sum_\mu \overline{u}_\mu^2(i)/D} \tag{116}$$

and the full inverse is given by

$$\Lambda_{\alpha\beta}(ij)=\delta_{ij}\left(D^{-1}\delta_{\alpha\beta}-\frac{\overline{u}_\alpha(i)\overline{u}_\beta(i)}{D^2\left[1+\sum_\mu \overline{u}_\mu^2(i)/D\right]}\right). \tag{117}$$

With these results it is easy to see that the quantity J_4 can be written in the simple form

$$J_4=(n!)^2\det\Lambda(11)\det\Lambda(22), \tag{118}$$

$$\det W=\det W(11)\det W(22), \tag{119}$$

and

$$\det\Lambda(ii)=\frac{1}{\det W(ii)}. \tag{120}$$

It is then easy to see that

$$\det W[ii]=D^n[1+v_i^2/\overline{v^2}], \tag{121}$$

where $\overline{v^2}$ is given by Eq. (29). If we then carefully keep track of all the factors we see that Eq. (108) reduces to

$$\lim_{x\to\infty}C_{nn}P[\vec{v}_1,\vec{v}_2]=n_0P[\vec{v}_1]n_0P[\vec{v}_2], \tag{122}$$

where $n_0P[\vec{v}_i]$ is given by Eq. (30) in Ref. [6] and, after proper normalization, leads to the expression for the single-vortex velocity probability distribution given by Eq. (28).

C. $n=d=2$ case

The general expression for $C_{nn}P[\vec{v}_1,\vec{v}_2]$ is complicated. Let us restrict ourselves here to the case of $n=d=2$ where det$W$ and J_4 can be evaluated explicitly. Let us define

$$\det A(ij)=A_{11}(ij)A_{22}(ij)-A_{12}(ij)A_{21}(ij), \tag{123}$$

where the matrix $A_{\alpha\beta}(ij)$ is either W or its inverse Λ. We can also define

$$\begin{aligned}Q_A=&[A_{11}(22)A_{21}(21)-A_{21}(22)A_{11}(21)][A_{12}(11)A_{22}(12)\\&-A_{22}(11)A_{12}(12)]-[A_{12}(22)A_{21}(21)\\&-A_{22}(22)A_{11}(21)][A_{12}(11)A_{21}(12)-A_{22}(11)A_{11}(12)]\\&-[A_{11}(22)A_{22}(21)-A_{21}(22)A_{12}(21)][A_{11}(11)A_{22}(12)\\&-A_{21}(11)A_{12}(12)]+[A_{12}(22)A_{22}(21)\\&-A_{22}(22)A_{12}(21)][A_{11}(11)A_{21}(12)\\&-A_{21}(11)A_{11}(12)].\end{aligned} \tag{124}$$

In terms of these quantities we have

$$\det W=\det W(11)\det W(22)+\det W(12)\det W(21)+Q_W \tag{125}$$

and

$$J_4=4[\det\Lambda(11)\det\Lambda(22)+3\det\Lambda(12)\det\Lambda(21)+Q_\Lambda]. \tag{126}$$

It is clear that the last nontrivial step before evaluating $C_{nn}P[\vec{v}_1,\vec{v}_2]$ is to determine $\Lambda_{\alpha\beta}(ij)$. This will be carried out in general in Sec. IV E; however, most of the important physics can be extracted in the problem by considering the simple case where the transverse velocities are both zero. In this case one can make substantial analytical progress.

D. Zero transverse velocities

Before tackling the complete determination of $P[\vec{v}_1,\vec{v}_1]$ it is very instructive to study the much simpler case where the transverse velocities are set to zero. This case is of interest not only because it is simple but also because it is the most probable situation. The most likely situation is that each of the two tagged vortices will have zero transverse velocity.

If the transverse velocities are zero then the problem simplifies since the matrix W reduces to the diagonal form

$$W_{\alpha\beta}(ij)=\delta_{\alpha\beta}D_{ij}^{\alpha}, \tag{127}$$

where

$$D_{ij}^{\alpha}=d_{ij}^{\alpha}+\delta_{\alpha,L}[u(i)\Omega_{ij}^{(1)}u(j)+u(i)\Omega_{ij}^{(2)}\eta_j+\eta_i\Omega_{ij}^{(2)}u(j)] \tag{128}$$

and $u(i)=u_L(i)$. Clearly, the longitudinal and transverse degrees of freedom are uncoupled and we have after some manipulations

$$D_{ij}^{T}=d^{T}[\delta_{ij}+f_T\delta_{j,i+1}], \tag{129}$$

while

$$D_{ij}^{L}=a_i\delta_{ij}+b\,\delta_{j,i+1}, \tag{130}$$

where

$$a_i=(M^{-1})_{55}+(M^{-1})_{33}\overline{u(i)}^2-2\,\eta_i(M^{-1})_{35}\overline{u(i)}, \tag{131}$$

$$b=(M^{-1})_{56}+(M^{-1})_{34}\overline{u(1)}\,\overline{u(2)}+(M^{-1})_{36}[\overline{u(2)}-\overline{u(1)}], \tag{132}$$

and

$$\overline{u(i)}=\frac{u(i)}{\Gamma c}. \tag{133}$$

In this case we see that W is a relatively simple matrix. We need its determinant and then its inverse on the way to evaluating the quantity J_4. The matrix of W for general n is simply given by

$$\det W=(\det D^{T})^{n-1}(\det D^{L}), \tag{134}$$

where

$$\det D^{T}=(d^{T})^2(1-f_T^2)=\frac{1}{S_T^2-C_T^2} \tag{135}$$

and

$$\det D^{L}=a_1a_2-b^2. \tag{136}$$

The quantity $\det D^{L}$ is key in the development and we shall return to it soon. First we need to evaluate the inverse of W to complete the calculation. In this case this involves the solution of the equation

$$\sum_k D_{ik}^{\alpha}\Lambda_{\alpha\beta}(kj)=\delta_{\alpha\beta}\delta_{ij}, \tag{137}$$

which is easily found to be given by

$$\Lambda_{\alpha\beta}(ij)=\delta_{\alpha\beta}[\delta_{\alpha,L}\Lambda_L(ij)+\delta_{\alpha,T}\Lambda_T(ij)], \tag{138}$$

where

$$\Lambda_L(ij)=\frac{1}{\det D^{L}}[a_{i+1}\delta_{ij}-b\,\delta_{j,i+1}] \tag{139}$$

and

$$\Lambda_T(ij)=\frac{1}{\det D^{T}}d^{T}[\delta_{ij}-f_T\delta_{j,i+1}]. \tag{140}$$

Using these results, one can work out the quantity J_4 for the case $n=d=2$ with the result

$$J_4=4\{\det\Lambda(11)\det\Lambda(22)+3[\det\Lambda(22)]^2-\Lambda_L(22)\Lambda_L(11)\Lambda_T^2(12)-\Lambda_T(22)\Lambda_T(11)\Lambda_L^2(12)\}, \tag{141}$$

where we have used the fact that the matrices $\Lambda_L(ij)$ and $\Lambda_T(ij)$ are symmetric. Putting in the explicit forms for Λ, we obtain

$$J_4=\frac{4}{(\det D^{T})^2(\det D^{L})^2}[\det D^{L}\det D^{T}+2b^2(d^{T}f_T)^2]. \tag{142}$$

Putting all of this together for $n=d=2$ and the transverse velocities zero, we have

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\frac{4}{(\Gamma c)^6}\frac{1}{\det M}\left[\frac{1}{(\det D^{L})^2\det D^{T}}+\frac{2b^2C_T^2}{(\det D^{L})^3}\right]. \tag{143}$$

Clearly, the next step is the explicit evaluation of $\det D^{L}$. Using the expressions for a_i and b given by Eqs. (131) and (132), we obtain, after some algebra, that

$$\det D^{L}=\frac{\sigma^4}{D_OD_E}\bigg[\overline{\gamma_0}+\frac{\overline{\gamma_1}}{2}[\tilde{u}(1)-\tilde{u}(2)]+\overline{\gamma_A}[\tilde{u}^2(1)+\tilde{u}^2(2)]+\overline{\gamma_B}\tilde{u}(1)\tilde{u}(2)+\frac{\overline{\gamma_3}}{2}\tilde{u}(1)\tilde{u}(2)[\tilde{u}(1)-\tilde{u}(2)]+\overline{\gamma_4}\tilde{u}^2(1)\tilde{u}^2(2)\bigg], \tag{144}$$

where the scaled longitudinal velocities are defined by

$$\tilde{u}(i)=\overline{u(i)}L \tag{145}$$

and

$$\overline{\gamma_0}=-\kappa_O^{(2)}\kappa_E^{(2)}, \tag{146}$$

$$\overline{\gamma_1}=2[(1-f)\kappa_O^{(1)}\kappa_E^{(2)}+(1+f)\kappa_E^{(1)}\kappa_O^{(2)}], \tag{147}$$

$$\overline{\gamma_A}=(1-f)\frac{D_E}{8\sigma^3}+(1+f)\frac{D_O}{2\sigma^3}+\frac{\overline{\gamma_2}}{4}, \tag{148}$$

$$\overline{\gamma_B}=(1-f)\frac{D_E}{4\sigma^3}+(1+f)\frac{D_O}{\sigma^3}-\frac{\overline{\gamma_2}}{2}, \tag{149}$$

$$\overline{\gamma_2}=(1-f)\kappa_O^{(0)}\kappa_E^{(2)}-(1+f)\kappa_E^{(0)}\kappa_O^{(2)}-4(1-f^2)\kappa_O^{(1)}\kappa_E^{(1)}, \tag{150}$$

$$\overline{\gamma_3}=2(1-f^2)[\kappa_O^{(1)}\kappa_E^{(0)}-\kappa_E^{(1)}\kappa_O^{(0)}], \tag{151}$$

$$\overline{\gamma_4}=(1-f^2)\kappa_E^{(0)}\kappa_O^{(0)}, \tag{152}$$

where the κ's are given as functions of f by Eqs. (A91)–(A96) in the Appendix. D_O and D_E are given as functions of f by Eqs. (A83) and (A84) in the Appendix. Note the relationship $2\overline{\gamma_A}-\overline{\gamma_B}=\overline{\gamma_2}$. Notice the crucial result that after rescaling the velocities by a factor of L, the time dependence drops out of $\det D^L$. This will eventually lead to the result that scaling holds for the probability distribution at late times if we rescale velocities in this manner.

We also need to express the quantity b in terms of the κ's. It is convenient to write

$$b=b_O+b_E, \tag{153}$$

where

$$b_O=\frac{\sigma^2}{4D_O}\{-\kappa_O^{(2)}-(1-f)\kappa_O^{(0)}\tilde{u}(1)\tilde{u}(2)+(1-f)\kappa_O^{(1)}\times[\tilde{u}(2)-\tilde{u}(1)]\}, \tag{154}$$

$$b_E=\frac{\sigma^2}{D_E}\{-\kappa_E^{(2)}+(1+f)\kappa_E^{(0)}\tilde{u}(1)\tilde{u}(2)+(1+f)\kappa_E^{(1)}\times[\tilde{u}(2)-\tilde{u}(1)]\}. \tag{155}$$

The last ingredient needed to evaluate the probability distribution is

$$C_T=-\frac{C_0'}{R}=-\sigma\frac{f'(x)}{x}=\sigma f(x), \tag{156}$$

using the OJK form for $f(x)$ in the last step.

Let us look first at the small-x limit. Since the OJK form for $f(x)$ is easily expanded in a power series in x and we can extract to leading order in x, $\overline{\gamma_0}=32x^4$, $\overline{\gamma_1}=-24x^5$, $\overline{\gamma_2}=\frac{26}{3}x^6$, $\overline{\gamma_3}=-\frac{4}{3}x^7$, $\overline{\gamma_4}=x^8/12$, $\overline{\gamma_A}=\frac{3}{8}x^2$, and $\overline{\gamma_B}=2\overline{\gamma_A}$. We also need

$$D_E=16x^2\sigma^3, \tag{157}$$

$$D_O=\sigma^3\frac{x^8}{48}. \tag{158}$$

Notice that the $\overline{\gamma_A}$ and $\overline{\gamma_B}$ dominate the expression for $\det D^L$ in the small-x limit and we can write to leading orders in x

$$\det D^L=\frac{\sigma^4}{D_OD_E}\{\overline{\gamma_0}+\overline{\gamma_A}[\tilde{u}(1)+\tilde{u}(2)]^2\} \tag{159}$$

plus terms that are higher order in x. We can write this in the more convenient form

$$\det D^L=\frac{\sigma^4\overline{\gamma_0}}{D_OD_E}[1+V^2/V_0^2], \tag{160}$$

where after some algebra we obtain

$$\frac{\sigma^4\overline{\gamma_0}}{D_OD_E}=\frac{96}{\sigma^6x^6}, \tag{161}$$

$$V=\tilde{u}(1)+\tilde{u}(2), \tag{162}$$

and

$$V_0^2=\frac{\overline{\gamma_0}}{\overline{\gamma_A}}=8x^2. \tag{163}$$

Similarly, we find for small x that b is dominated by b_0 and given to leading order in x by $b=-384/x^4\sigma$. We also need $C_T=\sigma$ to leading order in x. The probability distribution is dominated in the small-x limit by the term proportional to b^2. The other term is down by a factor of x^4. Putting all of this together, we obtain

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\left(\frac{\sigma^2}{\Gamma c}\right)^6\frac{1}{(1+V^2/V_0^2)^3}. \tag{164}$$

Then, as $x\rightarrow 0$ we find, with increasing probability, that

$$V=\tilde{u}(1)+\tilde{u}(2)=0. \tag{165}$$

This is just the physical statement that there is very low probability that there is a nonzero momentum of the center of mass (c.m.) of the two tagged vortices. Thus all of the action is *in* the center of mass where we can set

$$\tilde{u}(2)=-\tilde{u}(1)\equiv\tilde{u}. \tag{166}$$

If we return to the probability distribution for the case where the c.m. momentum is zero, then

$$\det D^L=\frac{\sigma^4}{D_OD_E}[\overline{\gamma_0}+\overline{\gamma_1}\tilde{u}+\overline{\gamma_2}\tilde{u}^2+\overline{\gamma_3}\tilde{u}^3+\overline{\gamma_4}\tilde{u}^4] \tag{167}$$

and

$$b_O=\frac{\sigma^2}{4D_O}[-\kappa_O^{(2)}+(1-f)\kappa_O^{(0)}\tilde{u}^2+2(1-f)\kappa_O^{(1)}\tilde{u}], \tag{168}$$

$$b_E=\frac{\sigma^2}{D_E}[-\kappa_E^{(2)}-(1+f)\kappa_E^{(0)}\tilde{u}^2+2(1+f)\kappa_E^{(1)}\tilde{u}]. \tag{169}$$

In the small-x limit these reduce to

$$b=b_O+b_E=-\frac{384}{x^4\sigma}f_\beta, \tag{170}$$

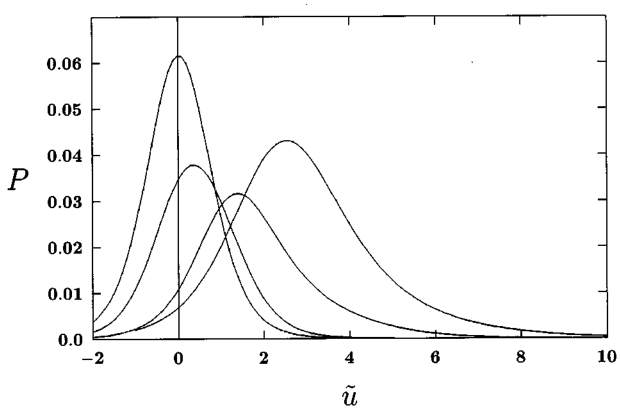

FIG. 3. Plot of unnormalized probability $P[\vec{v}_1,\vec{v}_2]$ for different values of the scaled distance x between the two tagged vortices versus the scaled velocity in the center of mass $\tilde{u}=\hat{x}\cdot\vec{u}(2)L/\Gamma c=-\hat{x}\cdot\vec{u}(1)L/\Gamma c$. The normalization changes with x so the different heights of the curves are not significant in this plot. The curves, as one moves from left to right, are labeled by x=3.0, 2.0, 1.0, and 0.7, respectively.

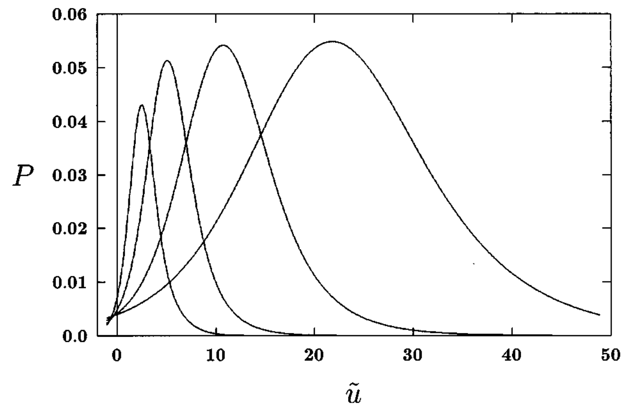

FIG. 4. Same as Fig. 3, except x=0.7, 0.4, 0.2, and 0.1 as one moves from left to right.

where

$$f_\beta=1-\frac{u}{4}+\frac{u^2}{48}, \tag{171}$$

$$\det(D^L)=\frac{96}{\sigma^6x^6}f_L, \tag{172}$$

with

$$f_L=1-\frac{3}{4}u+\frac{13}{48}u^2-\frac{1}{24}u^3+\frac{1}{384}u^4, \tag{173}$$

and the scaled velocity is given by

$$u=\tilde{u}x. \tag{174}$$

The probability distribution is given in the $x\rightarrow 0$ limit by

$$C_{nn}P[\vec{v}_1,\vec{v}_2]=\left(\frac{\sigma^2}{\Gamma c}\right)^6\frac{f_\beta^2}{f_L^3}. \tag{175}$$

A key conclusion we can draw at this point is that it is only the combination $u=\tilde{u}x$ that enters the probability distribution with high probability as $x\rightarrow 0$. Thus the relative velocity increases as $1/x$ as $x\rightarrow 0$. We plot f_β^2/f_L^3 as a function of u in Fig. 2. We note that the most probable values of the relative velocity as a function of x for small x are given by

$$v_L\equiv\frac{\Gamma c}{L}\frac{\kappa}{x}=\frac{\Gamma c\kappa}{R}, \tag{176}$$

with $\kappa=2.19$ a pure number. This is the result quoted in the Introduction.

If we then plot the two-vortex velocity probability distribution function for zero transverse velocities in the c.m. for general x as shown in Figs. 3 and 4, we obtain the most probable relative velocities as a function of x as shown in Fig. 1. The interpretation that this is the interaction between vortices and antivortices holds only out to modest values of x where the population of same-signed vortices begins to appear (see Ref. [5]).

E. General evaluation

The complete determination of the two-vortex velocity probability distribution as a general function of $\vec{v}_1$ and $\vec{v}_2$ can be carried out in the $n=d=2$ case if one can invert the matrix $W_{\alpha\beta}(ij)$ to obtain its inverse $\Lambda_{\alpha\beta}(ij)$ defined by

$$\sum_{\mu,k}W_{\alpha\mu}(ik)\Lambda_{\mu\beta}(kj)=\delta_{\alpha\beta}\delta_{ij}. \tag{177}$$

This inversion is a quite unpleasant task if one heads in the wrong direction. It is useful in order to make the development more transparent to introduce a mixed operator notation where $W_{\alpha\beta}$ is an operator in the space associated with the indices i and j,

$$W_{\alpha\beta}(ij)=\langle i|W_{\alpha\beta}|j\rangle. \tag{178}$$

Then the matrix $W_{\alpha\beta}$ is given by

$$W_{\alpha\beta}=\delta_{\alpha\beta}d^\alpha+u_\alpha\Omega^{(1)}u_\beta+u_\alpha\Omega^{(2)}L_\beta+L_\alpha\Omega^{(2)}u_\beta, \tag{179}$$

where u_α and L_α are diagonal operators

$$\langle i|u_\alpha|j\rangle=u_\alpha(i)\delta_{ij}, \tag{180}$$

$$\langle i|L_\alpha|j\rangle=L_\alpha(i)\delta_{ij}. \tag{181}$$

The key idea is that if we can write $W_{\alpha\beta}$ in the form

$$W_{\alpha\beta}=\delta_{\alpha\beta}D_\alpha+P_\alpha\tilde{P}_\beta, \tag{182}$$

where $\tilde{P}_\beta$ is the transpose of P_α, then we can carry out the inversion straight away. Let us first show this and then return to show that W can be written in the assumed form.

We want to invert the equation

$$\sum_\mu W_{\alpha\mu}\Lambda_{\mu\beta}=\delta_{\alpha\beta}. \tag{183}$$

Inserting the assumed form (182) for W we obtain

$$D^{\alpha}\Lambda_{\alpha\beta}+P_{\alpha}\sum_{\mu}\tilde{P}_{\mu}\Lambda_{\mu\beta}=\delta_{\alpha\beta}. \tag{184}$$

Multiplying from the left by the matrix inverse of D_{α}, this becomes

$$\Lambda_{\alpha\beta}+D_{\alpha}^{-1}P_{\alpha}\sum_{\mu}\tilde{P}_{\mu}\Lambda_{\mu\beta}=D_{\alpha}^{-1}\delta_{\alpha\beta}. \tag{185}$$

This equation is then in the form of a trap for the quantity $\Sigma_{\mu}\tilde{P}_{\mu}\Lambda_{\mu\beta}$. Multiplying Eq. (185) by $\tilde{P}_{\alpha}$ and summing over α, we obtain a closed equation for $\Sigma_{\mu}\tilde{P}_{\mu}\Lambda_{\mu\beta}$, which has the solution

$$\sum_{\mu}\tilde{P}_{\mu}\Lambda_{\mu\beta}=[1+Q]^{-1}\sum_{\mu}\tilde{P}_{\beta}D_{\beta}^{-1}, \tag{186}$$

where the 2×2 symmetric matrix Q is defined by

$$Q=\sum_{\mu}\tilde{P}_{\mu}D_{\mu}^{-1}P_{\mu}. \tag{187}$$

This leads directly to the final result

$$\Lambda_{\alpha\beta}=D_{\alpha}^{-1}\delta_{\alpha\beta}-D_{\alpha}^{-1}P_{\alpha}[1+Q]^{-1}\tilde{P}_{\beta}D_{\beta}^{-1}, \tag{188}$$

which is clearly symmetric. This gives a practical expression for the inverse once one has identified the matrices D and P.

The key observation that allows one to write W in the desired form given by Eq. (182) is that the matrix $\Omega_{ij}^{(1)}$ can be factorized in the form

$$\Omega_{ij}^{(1)}=\sum_{k}\omega_{ik}\tilde{\omega}_{kj}, \tag{189}$$

where

$$\omega_{ij}=\omega_0\delta_{ij}+\omega_1\delta_{j,i+1} \tag{190}$$

and

$$\omega_0=\frac{\sqrt{S_b}}{2\Gamma c}[\sqrt{1+f_b}+\sqrt{1-f_b}], \tag{191}$$

$$\omega_1=\frac{\sqrt{S_b}}{2\Gamma c}[\sqrt{1+f_b}-\sqrt{1-f_b}]. \tag{192}$$

Using this factorization result, it is then easy to show that W can be written in the form (182) with

$$D_{ij}^{\alpha}=d_{ij}^{\alpha}-\delta_{\alpha,L}\sum_{k}C_L(ik)\tilde{C}_L(kj), \tag{193}$$

$$P_{\alpha}(ij)=u_{\alpha}(i)\omega_{ij}+C_{\alpha}(ij), \tag{194}$$

with

$$C_{\alpha}(ij)=L_{\alpha}(i)\sum_{k}\Omega_{ik}^{(2)}\omega_{kj}^{-1}. \tag{195}$$

Combining these results, one has an explicit expression for the two-vortex velocity probability distribution for arbitrary velocities. The major qualitative feature of including the transverse velocities is to allow one to look at the widths of the distributions in the transverse directions since we find the most probable configurations are those where the transverse velocities of both vortices are zero. These widths turn out to be comparable to those associated with the longitudinal modes.

V. DISCUSSION

In the analysis here we have looked at the correlation between vortices regardless of their signs. At short relative distances, where it is unlikely to have two vortices of the same sign, one can interpret the results in terms of vortex-antivortex dynamics. It is clear that one can go further, as discussed by Mazenko and Wickham [5], and separate the probability distribution into that for vortex-vortex and vortex-antivortex pairs. The key idea, which is essentially equivalent to that used in the case of spatial correlations, is that a factor of $\mathcal{P}_{+}(1)\equiv\frac{1}{2}[1+\mathrm{sgn}\mathcal{D}(1)]$ restricts one to the positive-charge vortex sector, while $\mathcal{P}_{-}(1)=\frac{1}{2}[1-\mathrm{sgn}\mathcal{D}(1)]$ restricts one to the negative-charge antivortex sector. Thus the probability for vortex-vortex correlations is

$$C_{vv}P_{vv}(12)=\langle n(1)\delta(\vec{v}_1-\vec{v}(1))\mathcal{P}_{+}(1)n(2)\times\delta(\vec{v}_2-\vec{v}(2))\mathcal{P}_{+}(2)\rangle. \tag{196}$$

The vortex-antivortex contribution is given by

$$C_{av}P_{av}(12)=\langle n(1)\delta(\vec{v}_1-\vec{v}(1))\mathcal{P}_{-}(1)n(2)\times\delta(\vec{v}_2-\vec{v}(2))\mathcal{P}_{+}(2)\rangle. \tag{197}$$

These quantities can be multiplied out and, and after using symmetry to show that the correlation between the signed and unsigned quantities is zero, can be expressed in terms of the probability distribution determined in this paper and

$$C_{\rho\rho}P_{\rho\rho}(12)=\langle\rho(1)\delta(\vec{v}_1-\vec{v}(1))\rho(2)\delta(\vec{v}_2-\vec{v}(2))\rangle, \tag{198}$$

which has not yet been computed. It is expected that $P_{\rho\rho}(12)$ will be difficult to determine because of the addition factors of the sgn $\mathcal{D}$. The analysis will be essentially identical in structure up to Eq. (125) with the expression for J_2 showing the replacement

$$\mathcal{D}^2(t(i))\rightarrow\mathcal{D}(t(i))|\mathcal{D}(t(i))|. \tag{199}$$

The resulting integral for J_2 cannot then be represented in the product form given by Eq. (103). This remains a problem to be solved.

In principle, Eq. (108) gives an expression that can be integrated over all velocities to give C_u and determine the overall normalization. It is not clear how to do this analytically since the velocities appear in a complicated fashion in $\det W$ and J_4. A numerical determination is quite feasible.

VI. CONCLUSION

In this paper we have shown how one can make progress in an analysis of the dynamics of point vortices in the context of phase ordering kinetics. The results include the effects of other vortices and order-parameter fluctuations on the dynamics of the tagged vortices. The results appear completely physical and the determination of the relative velocity as short distances appears to be a useful result. The method used here appears to generalize easily to the case of string defects. This will be the subject of subsequent work.

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APPENDIX: GAUSSIAN AVERAGE

In this Appendix we work out the Gaussian average

$$G_2(\xi,\vec{b})=\langle W_1[\xi,\vec{b}]W_2[\xi,\vec{b}]\rangle, \quad \text{(A1)}$$

where the W's are defined by

$$W_i[\xi,\vec{b}]\equiv\delta(\vec{m}(i))\delta(\vec{b}(i)-\nabla_i^2\vec{m}(i)) \times\prod_{\mu,\nu=1}^{n}\delta(\xi_\mu^\nu(i)-\nabla_\mu^{(i)}m_\nu(i)), \quad \text{(A2)}$$

where we have already assumed that $n=d$ in the product. The first step in the evaluation of G_2 is to use the Fourier representation for the δ function to obtain

$$W_i[\xi,\vec{b}]=\int d\tilde{\Omega}[i]e^{i\vec{q}_i\cdot\vec{m}(i)}e^{-i\vec{s}_i\cdot[\vec{b}(i)-\nabla_i^2\vec{m}(i)]} \times\exp\left(-i\sum_{\mu,\nu}k_\mu^\nu(i)[\xi_\mu^\nu(i)-\nabla_\mu m_\nu(i)]\right), \quad \text{(A3)}$$

where we have defined

$$d\tilde{\Omega}[i]=\frac{d^nq_i}{(2\pi)^n}\frac{d^ns_i}{(2\pi)^n}\prod_{\mu,\nu=1}^{n}\left[\frac{dk_\mu^\nu(i)}{2\pi}\right]. \quad \text{(A4)}$$

We can rewrite this in the more useful form

$$W_i[\xi,\vec{b}]=\int d\tilde{\Omega}[i]\exp\left[-i\left(\vec{s}_i\cdot\vec{b}(i)+\sum_{\mu,\nu=1}^{n}k_\mu^\nu(i)\xi_\mu^\nu(i)\right)\right] \times\exp\left(\sum_{\alpha=1}^{n}\int d\overline{1}H_i^\alpha(\overline{1})m_\alpha(\overline{1})\right), \quad \text{(A5)}$$

where

$$\int d\overline{1}=\int d^d\overline{r}_1 d\overline{t}_1, \quad \text{(A6)}$$

$$H_i^\alpha(\overline{1})=i\left[q_i^\alpha+s_i^\alpha\nabla_{(i)}^2+\sum_{\mu=1}^{n}k_\mu^\alpha(i)\nabla_{(i)}^\mu\right]\delta(\overline{1i}), \quad \text{(A7)}$$

and $\nabla_{(i)}^\mu$ is the μth component of the gradient acting on $\vec{r}_i$. The average of interest can then be written as

$$G_2(\xi,\vec{b})=\int d\tilde{\Omega}[1]d\tilde{\Omega}[2]\exp\left[-i\sum_{i=1}^{2}\left(\vec{s}_i\cdot\vec{b}(i)+\sum_{\mu,\nu=1}^{n}k_\mu^\nu(i)\xi_\mu^\nu(i)\right)\right] \times\left\langle\exp\left[\sum_{i=1}^{2}\sum_{\alpha=1}^{n}\int d\overline{1}H_i^\alpha(\overline{1})m_\alpha(\overline{1})\right]\right\rangle. \quad \text{(A8)}$$

The average is of the standard form for a Gaussian average with the result

$$\left\langle\exp\left[\sum_{i=1}^{2}\sum_{\alpha=1}^{n}\int d\overline{1}H_i^\alpha(\overline{1})m_\alpha(\overline{1})\right]\right\rangle=e^{-(1/2)A_0}, \quad \text{(A9)}$$

where

$$A_0=-\sum_{i,j=1}^{2}\sum_{\alpha,\beta=1}^{n}\int d\overline{1}\int d\overline{2}H_i^\alpha(\overline{1})H_j^\beta(\overline{2})C_0(\overline{1,2})\delta_{\alpha,\beta} \quad \text{(A10)}$$

and we have used

$$\langle m_\alpha(\overline{1})m_\beta(\overline{2})\rangle=C_0(\overline{1,2})\delta_{\alpha,\beta}. \quad \text{(A11)}$$

Inserting the expression for H into A_0 we need the definitions

$$C_0(ii)\equiv S_0, \quad \text{(A12)}$$

$$[\nabla_{(i)}^2C_0(ij)]|_{i=j}\equiv-nS^{(2)}, \quad \text{(A13)}$$

$$[\nabla_{(j)}^2\nabla_{(i)}^2C_0(ij)]|_{i=j}\equiv S^{(4)}, \quad \text{(A14)}$$

$$[\nabla_{(j)}^\nu\nabla_{(i)}^\mu C_0(ij)]|_{i=j}=\delta_{\mu\nu}S^{(2)}. \quad \text{(A15)}$$

Using the fact that $C_0(12)$ depends only on the magnitude of $\vec{R}=\vec{r}_1-\vec{r}_2$, we convert all derivatives to those with respect to $\vec{R}$:

$$\nabla_{(1)}^\mu C_0(12)=C_0'\hat{R}_\mu, \quad \text{(A16)}$$

$$\nabla_{(2)}^\mu C_0(12)=-C_0'\hat{R}_\mu, \quad \text{(A17)}$$

where the prime indicates a derivative with respect to R. Going further, for $i=1$ and 2,

$$\nabla_{(i)}^2C_0(12)=\nabla_R^2C_0(R)=C_0''+\frac{(d-1)}{R}C_0', \quad \text{(A18)}$$

$$\nabla^{\mu}_{(1)}\nabla^{\nu}_{(2)}C_0(12)=-C_0''\hat{R}_\mu\hat{R}_\nu-\frac{C_0'}{R}(\delta_{\mu\nu}-\hat{R}_\mu\hat{R}_\nu), \tag{A19}$$

$$\nabla^2_{(2)}\nabla^{\mu}_{(2)}C_0(12)=-p\hat{R}_\mu, \tag{A20}$$

$$\nabla^2_{(1)}\nabla^{\mu}_{(1)}C_0(12)=p\hat{R}_\mu, \tag{A21}$$

where

$$p=C_0'''+\frac{(d-1)}{R}\left(C_0''-\frac{C_0'}{R}\right)=[\nabla^2_R C_0(R)]'. \tag{A22}$$

We see that it is then natural to use the coordinate system parallel and orthogonal to $\vec{R}$. Indeed, we can introduce the orthonormal set $\hat{R}^{\alpha}_{\beta}$ where

$$\sum_{\alpha=1}^{n}\hat{R}^{\mu}_{\alpha}\hat{R}^{\nu}_{\alpha}=\delta_{\mu\nu}, \tag{A23}$$

$$\sum_{\mu=1}^{n}\hat{R}^{\mu}_{\alpha}\hat{R}^{\mu}_{\beta}=\delta_{\alpha\beta}. \tag{A24}$$

The only other thing we need to know about this set is that

$$\hat{R}^1_\beta=\hat{R}_\beta. \tag{A25}$$

Next we define

$$W^{\alpha}_{\beta}(i)=\sum_{\mu=1}^{n}\hat{R}^{\beta}_{\mu}k^{\alpha}_{\mu}(i), \tag{A26}$$

which can be inverted to give

$$k^{\alpha}_{\mu}(i)=\sum_{\beta=1}^{n}\hat{R}^{\beta}_{\mu}W^{\alpha}_{\beta}(i). \tag{A27}$$

In terms of this new set of variables,

$$\sum_{i,\mu,\nu}[k^{\nu}_{\mu}(i)]^2=\sum_{i,\mu,\nu}\left(\sum_{\beta}\hat{R}^{\beta}_{\mu}W^{\nu}_{\beta}(i)\right)\left(\sum_{\sigma}\hat{R}^{\sigma}_{\mu}W^{\nu}_{\sigma}(i)\right)=\sum_{i,\beta,\nu}[W^{\nu}_{\beta}(i)]^2. \tag{A28}$$

We then have

$$A_0=A(\vec{q})+A(\vec{s})+A(\vec{W}_L)+A_T(W)+A_c(\vec{q},\vec{s},\vec{W}_L), \tag{A29}$$

where

$$A(\vec{q})=S_0\sum_{i=1}^{2}\vec{q}_i^{\,2}+2C_0\vec{q}_1\cdot\vec{q}_2, \tag{A30}$$

$$A(\vec{s})=S^{(4)}\sum_{i=1}^{2}\vec{s}_i^{\,2}+2(\nabla^4C_0)\vec{s}_1\cdot\vec{s}_2, \tag{A31}$$

$$A_L(\vec{W})=S^{(2)}\sum_{i=1}^{2}\vec{W}_L(i)^2-2(C_0'')\vec{W}_L(1)\cdot\vec{W}_L(2), \tag{A32}$$

where the longitudinal part of the tensor W is defined by

$$W^{\alpha}_{L}(i)=\sum_{\mu=1}^{n}\hat{R}^1_\mu k^{\alpha}_{\mu}(i). \tag{A33}$$

The *transverse* contribution to A_0 is given by

$$A_T(W)=\sum_{\mu=2}^{n}\sum_{\nu=1}^{n}\left[S^{(2)}\sum_{i=1}^{2}[W^{\nu}_{\mu}(i)]^2-2(C_0'/R)W^{\nu}_{\mu}(1)W^{\nu}_{\mu}(2)\right]. \tag{A34}$$

The term *coupling* the set $\vec{q},\vec{s},\vec{W}_L$ is given by

$$A_c(\vec{q},\vec{s},\vec{W}_L)=-2nS^{(2)}\sum_{i=1}^{2}\vec{q}_i\cdot\vec{s}_i+2(\vec{q}_1\cdot\vec{s}_2+\vec{q}_2\cdot\vec{s}_1)\nabla^2C_0-2(p\vec{s}_1+C_0'\vec{q}_1)\cdot\vec{W}_L(2)+2(p\vec{s}_2+C_0'\vec{q}_2)\cdot\vec{W}_L(1). \tag{A35}$$

Notice that the *transverse* modes decouple from the longitudinal set coupled in A_c.

It will be very useful for us to rewrite A_0 as a sum of a transverse part, already written down, and a longitudinal part that is a quadratic form in the vector

$$\vec{\phi}_\alpha=[\vec{q}(1),\vec{q}(2),\vec{s}(1),\vec{s}(2),\vec{W}_L(1),\vec{W}_L(2)], \tag{A36}$$

where we assume that the subscript α runs from 1 to 6. We have then

$$A_L=\sum_{\alpha,\beta}M_{\alpha\beta}\vec{\phi}_\alpha\cdot\vec{\phi}_\beta, \tag{A37}$$

where the matrix M is given explicitly by

$$M=\begin{pmatrix} S_0 & C_0 & u_1 & u_2 & 0 & u_4\\ C_0 & S_0 & u_2 & u_1 & -u_4 & 0\\ u_1 & u_2 & S^{(4)} & C_4 & 0 & u_3\\ u_2 & u_1 & C_4 & S^{(4)} & -u_3 & 0\\ 0 & -u_4 & 0 & -u_3 & S^{(2)} & C_2\\ u_4 & 0 & u_3 & 0 & C_2 & S^{(2)} \end{pmatrix}. \tag{A38}$$

Various quantities entering the matrix M are defined by

$$C_4=\nabla^4C_0=\frac{\sigma}{L^2}\nabla^4_x f(x), \tag{A39}$$

$$S^{(4)}=\nabla^4C_0|_{R=0}=8\frac{\sigma}{L^2}, \tag{A40}$$

$$C_2 = -C_0'' = -\sigma f'', \tag{A41}$$

$$u_1 = -nS^{(2)} = -n\sigma, \tag{A42}$$

$$u_2 = \nabla^2 C_0 = \sigma \nabla_x^2 f(x), \tag{A43}$$

$$u_3 = -p = -\frac{\sigma}{L}[\nabla_x^2 f(x)]', \tag{A44}$$

$$u_4 = -C_0' = -L\sigma f'(x). \tag{A45}$$

The complete change of variable from k to W in Eq. (A26) requires noting that the Jacobian taking one from k to W is one. The argument of the exponential outside the average in Eq. (A26) can also be written in terms of the set $\vec{\phi}_\alpha$ and the transverse part of W. Then one has

$$\sum_{i=1}^{2}\left[\sum_{\mu,\nu=1}^{n} k_\mu^\nu(i)\xi_\mu^\nu(i) + \vec{s}_i\cdot\vec{b}(i)\right]$$

$$= \sum_{\alpha=1}^{6} \vec{h}_\alpha\cdot\vec{\phi}_\alpha + \sum_{i=1}^{2}\sum_{\nu=1}^{n}\sum_{\mu=2}^{n} W_\mu^\nu(i) t_\mu^\nu(i), \tag{A46}$$

where

$$\vec{h}_1 = \vec{h}_2 = 0, \tag{A47}$$

$$\vec{h}_3 = \vec{b}(1), \tag{A48}$$

$$\vec{h}_4 = \vec{b}(2), \tag{A49}$$

$$\vec{h}_5 = \vec{t}_L(1), \tag{A50}$$

$$\vec{h}_6 = \vec{t}_L(2), \tag{A51}$$

and

$$t_\mu^\nu(i) \equiv \sum_{\beta=1}^{n} R_\mu^\beta(i)\xi_\beta^\nu(i). \tag{A52}$$

We then have the result that G_2 factorizes into longitudinal and transverse components:

$$G_2(\xi,\vec{b}) = G_T(t_T)G_L(\vec{b},\vec{t}_L). \tag{A53}$$

First consider the transverse contribution given by

$$G_T(t_T) = \int\left[\prod_{i=1}^{2}\prod_{\nu=1}^{n}\prod_{\mu=2}^{n}\frac{dW_\mu^\nu(i)}{2\pi}\right]\exp\left[-\frac{1}{2}A_T(W_T) - i\sum_{i=1}^{2}\sum_{\nu=1}^{n}\sum_{\mu=2}^{n} t_\mu^\nu(i)W_\mu^\nu(i)\right]. \tag{A54}$$

This is a standard Gaussian integral that can be evaluated with the results

$$G_T(t_T) = \left(\frac{\gamma_T}{2\pi S_T}\right)^{n(n-1)}\exp-\frac{\gamma_T^2}{2S_T}\left[\sum_{\mu=2}^{n}\sum_{\nu=1}^{n}\left(\sum_{i=1}^{2}[t_\mu^\nu(i)]^2 - 2f_T t_\mu^\nu(1)t_\mu^\nu(2)\right)\right], \tag{A55}$$

where

$$S_T = S^{(2)}, \tag{A56}$$

$$C_T = -\frac{C_0'}{R}, \tag{A57}$$

$$f_T = \frac{C_T}{S_T}, \tag{A58}$$

and

$$\gamma_T^2 = (1-f_T^2)^{-1}. \tag{A59}$$

The longitudinal contribution to G_2 is also a standard Gaussian integral:

$$G_L(\vec{b},\vec{t}_L) = \int\prod_{i=1}^{2}\left[\prod_{\alpha=1}^{6}\frac{d^n\phi_\alpha(i)}{(2\pi)^n}\right]\exp\left(-i\sum_{\alpha=1}^{6}\vec{h}_\alpha\cdot\vec{\phi}_\alpha\right)\times\exp\left(-\frac{1}{2}\sum_{\alpha,\beta=1}^{6}M_{\alpha\beta}\vec{\phi}_\alpha\cdot\vec{\phi}_\beta\right)$$

$$= \frac{1}{(2\pi)^{3n}}\frac{1}{(\det M)^{n/2}}\times\exp\left(-\frac{1}{2}\sum_{\alpha,\beta=1}^{6}\vec{h}_\alpha\cdot\vec{h}_\beta(M^{-1})_{\alpha\beta}\right). \tag{A60}$$

Thus the determination of G_2 reduces to an evaluation of the inverse and determinant of the matrix M given above. If we multiply M from left and right by the matrix

$$Q = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}, \tag{A61}$$

then the new matrix

$$\tilde{M}_{\alpha\beta} = \sum_{\mu,\nu} Q_{\mu\alpha}Q_{\nu\beta}M_{\mu\nu} \tag{A62}$$

has the block diagonal form

$$\widetilde{M}=\begin{pmatrix} 2(S_0-C_0) & 0 & u_1-u_2 & 0 & u_4 & 0 \\ 0 & \frac{1}{2}(S_0+C_0) & 0 & u_1+u_2 & 0 & u_4 \\ u_1-u_2 & 0 & \frac{1}{2}(S^{(4)}-C_4) & 0 & \frac{1}{2}u_3 & 0 \\ 0 & u_1+u_2 & 0 & 2(S^{(4)}+C_4) & 0 & 2u_3 \\ u_4 & 0 & \frac{1}{2}u_3 & 0 & \frac{1}{2}(S^{(2)}+C_2) & 0 \\ 0 & u_4 & 0 & 2u_3 & 0 & 2(S^{(2)}-C_2) \end{pmatrix}. \tag{A63}$$

It is easy to see that the inverse for M can be expressed in terms of the inverse of $\widetilde{M}$ as

$$(M^{-1})_{\mu\nu}=\sum_{\alpha,\beta} Q_{\mu\alpha}Q_{\nu\beta}(\widetilde{M}^{-1})_{\alpha\beta}. \tag{A64}$$

Since $\widetilde{M}$ is block diagonal the evaluation of its determinant and inverse elements is straightforward. We find

$$\det\widetilde{M}=D_O D_E, \tag{A65}$$

where D_O is the determinant of the *odd* part of the matrix given by

$$\begin{pmatrix} 2(S_0-C_0) & u_1-u_2 & u_4 \\ u_1-u_2 & \frac{1}{2}(S^{(4)}-C_4) & \frac{1}{2}u_3 \\ u_4 & \frac{1}{2}u_3 & \frac{1}{2}(S^{(2)}+C_2) \end{pmatrix}, \tag{A66}$$

so

$$2D_O=(S_0-C_0)[(S^{(4)}-C_4)(S^{(2)}+C_2)-u_3^2]-(u_1-u_2)\times[(u_1-u_2)(S^{(2)}+C_2)-u_3u_4]+u_4[u_3(u_1-u_2)-u_4(S^{(4)}-C_4)]. \tag{A67}$$

D_E is the determinant of the *even* part of the matrix given by

$$\begin{pmatrix} \frac{1}{2}(S_0+C_0) & u_1+u_2 & u_4 \\ u_1+u_2 & 2(S^{(4)}+C_4) & 2u_3 \\ u_4 & 2u_3 & 2(S^{(2)}-C_2) \end{pmatrix}, \tag{A68}$$

where

$$D_E/2=(S_0+C_0)[(S^{(4)}+C_4)(S^{(2)}-C_2)-u_3^2]-(u_1+u_2)\times[(u_1+u_2)(S^{(2)}-C_2)-u_3u_4]+u_4[u_3(u_1+u_2)-u_4(S^{(4)}+C_4)]. \tag{A69}$$

Since it is easy to show that $\det Q=1$ we obtain

$$\det\widetilde{M}=\det Q\det M\det Q=\det M=D_E D_O. \tag{A70}$$

The needed inverses are given by

$$(\widetilde{M}^{-1})_{33}=\frac{(S_0-C_0)(S^{(2)}+C_2)-u_4^2}{D_O}, \tag{A71}$$

$$(\widetilde{M}^{-1})_{35}=\frac{(u_1-u_2)u_4-(S_0-C_0)u_3}{D_O}, \tag{A72}$$

$$(\widetilde{M}^{-1})_{55}=\frac{(S_0-C_0)(S^{(4)}-C_4)-(u_1-u_2)^2}{D_O}, \tag{A73}$$

$$(\widetilde{M}^{-1})_{44}=\frac{(S_0+C_0)(S^{(2)}-C_2)-u_4^2}{D_E}, \tag{A74}$$

$$(\widetilde{M}^{-1})_{46}=\frac{(u_1+u_2)u_4-(S_0+C_0)u_3}{D_E}, \tag{A75}$$

$$(\widetilde{M}^{-1})_{66}=\frac{(S_0+C_0)(S^{(4)}+C_4)-(u_1+u_2)^2}{D_E}. \tag{A76}$$

All odd-even inverse elements such as $(\widetilde{M}^{-1})_{34}$ vanish and the rest of the elements follow using the symmetry of $(\widetilde{M}^{-1})$. One can then easily extract the inverse elements of (M^{-1}):

$$(M^{-1})_{33}=(M^{-1})_{44}=\tfrac{1}{4}(\widetilde{M}^{-1})_{33}+(\widetilde{M}^{-1})_{44}, \tag{A77}$$

$$(M^{-1})_{55}=(M^{-1})_{66}=\tfrac{1}{4}(\widetilde{M}^{-1})_{55}+(\widetilde{M}^{-1})_{66}, \tag{A78}$$

$$(M^{-1})_{34}=(\widetilde{M}^{-1})_{44}-\tfrac{1}{4}(\widetilde{M}^{-1})_{33}, \tag{A79}$$

$$(M^{-1})_{35}=-(M^{-1})_{46}=\tfrac{1}{4}(\widetilde{M}^{-1})_{35}-(\widetilde{M}^{-1})_{46}, \tag{A80}$$

$$(M^{-1})_{36}=-(M^{-1})_{45}=\tfrac{1}{4}(\widetilde{M}^{-1})_{35}+(\widetilde{M}^{-1})_{46}, \tag{A81}$$

$$(M^{-1})_{56}=\tfrac{1}{4}(\widetilde{M}^{-1})_{55}-(\widetilde{M}^{-1})_{66}. \tag{A82}$$

In terms of f and its derivatives we find that we can write

$$D_E=2\sigma^3\{\kappa_E^{(0)}\kappa_E^{(2)}-(1+f)[\kappa_E^{(1)}]^2\}, \tag{A83}$$

$$D_O=\tfrac{1}{2}\sigma^3\{-\kappa_O^{(0)}\kappa_O^{(2)}-(1-f)[\kappa_O^{(1)}]^2\}, \tag{A84}$$

$$D_O(\widetilde{M}^{-1})_{33}=L^2\sigma^2(1-f)\kappa_O^{(0)}, \tag{A85}$$

$$D_O(\widetilde{M}^{-1})_{35}=L\sigma^2(1-f)\kappa_O^{(1)}, \tag{A86}$$

$$D_O(\widetilde{M}^{-1})_{55} = -\sigma^2 \kappa_O^{(2)}, \tag{A87}$$

$$D_E(\widetilde{M}^{-1})_{44} = L^2 \sigma^2 (1+f) \kappa_E^{(0)}, \tag{A88}$$

$$D_E(\widetilde{M}^{-1})_{46} = L \sigma^2 (1+f) \kappa_E^{(1)}, \tag{A89}$$

$$D_E(\widetilde{M}^{-1})_{66} = \sigma^2 \kappa_E^{(2)}. \tag{A90}$$

These results give the explicit L dependences of the various matrix elements. The κ's are independent of L and given by

$$\kappa_E^{(1)} = (\nabla^2 f)' - \frac{f'(\nabla^2 f)_+}{1+f}, \tag{A91}$$

$$\kappa_E^{(2)} = (1+f)(\nabla^4 f)_+ - [(\nabla^2 f)_+]^2, \tag{A92}$$

$$\kappa_E^{(0)} = (f'')_- - \frac{(f')^2}{1+f}, \tag{A93}$$

$$\kappa_O^{(1)} = (\nabla^2 f)' + \frac{f'(\nabla^2 f)_-}{1-f}, \tag{A94}$$

$$\kappa_O^{(2)} = (1-f)(\nabla^4 f)_- + [(\nabla^2 f)_-]^2, \tag{A95}$$

$$\kappa_O^{(0)} = -(f'')_+ - \frac{(f')^2}{1-f}, \tag{A96}$$

where we have introduced the notation

$$A_\pm = A(x) \pm A(0). \tag{A97}$$

[1] For a recent review of the role of defects in both a cosmological and condensed matter context see *Formation and Interactions of Topological Defects*, Vol. 349 of *NATO Advanced Study Institute Series B: Physics*, edited by A.-C. Davis and R. Brandenberger (Plenum, New York, 1995).

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